Homework 3 - Answers to Selected Questions

1. a. $f(x,y)=x^2+y$ is defined for all pairs (x,y) and by choosing y very large or small we can make $f(x,y)$ arbitrarily large or small. Therefore

$$\boxed{\begin{array}{l}\text{Domain} = \mathbb{R}^2 \text{ - all possible pairs of real numbers} \\ \text{Range} = \mathbb{R}\end{array}}$$

b. $g(x,y)=\ln(xy)$ is defined only when $xy>0$ and approaches ∞ as $xy\to\infty$, $-\infty$ as $xy\to 0^+$.

$$\boxed{\begin{array}{l}\text{Domain} = \{(x,y) : xy \geqslant 0\} \\ \text{Range} = \mathbb{R}\end{array}}$$

c. $h(x,y)=ye^x-2$ is defined for all (x,y). As $y\to\infty$, $h(x,y)\to\infty$ and as $y\to-\infty$, $h(x,y)\to-\infty$ for any fixed x.

$$\boxed{\begin{array}{l}\text{Domain} = \mathbb{R}^2 \\ \text{Range} = \mathbb{R}\end{array}}$$

d. $u(x,y)=x^y$ is defined only when $x\geqslant 0$ (for instance $(-1)^{1/2}$ or $(-1)^{\frac{1}{4}}$ is undefined). The function x^y can take any non-negative value.

$$\boxed{\begin{array}{l}\text{Domain} = \{(x,y) : x\geqslant 0\} \\ \text{Range} = [0,\infty)\end{array}}$$

2. a. At $z=0$

$\sqrt{4x^2+y^2}=0 \Rightarrow 4x^2+y^2=0$

$z=1$ ellipse with radii 1/2 and 1

$\sqrt{4x^2+y^2}=1 \Rightarrow 4x^2+y^2=1$

$z=2$

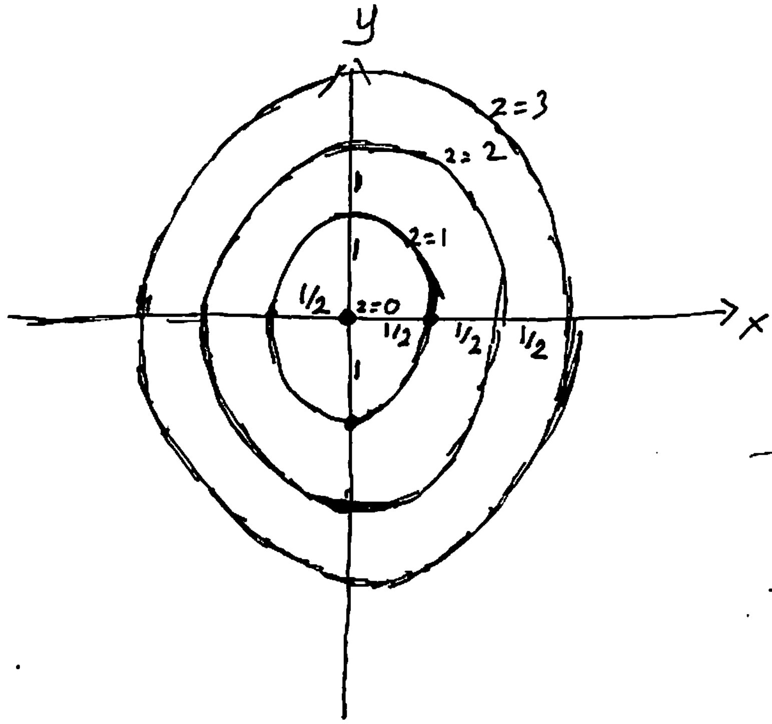
$\sqrt{4x^2+y^2}=2 \Rightarrow 4x^2+y^2=4$

ellipse with radii 1 and 2

$z=3$

$\sqrt{4x^2+y^2}=3 \Rightarrow 4x^2+y^2=9$

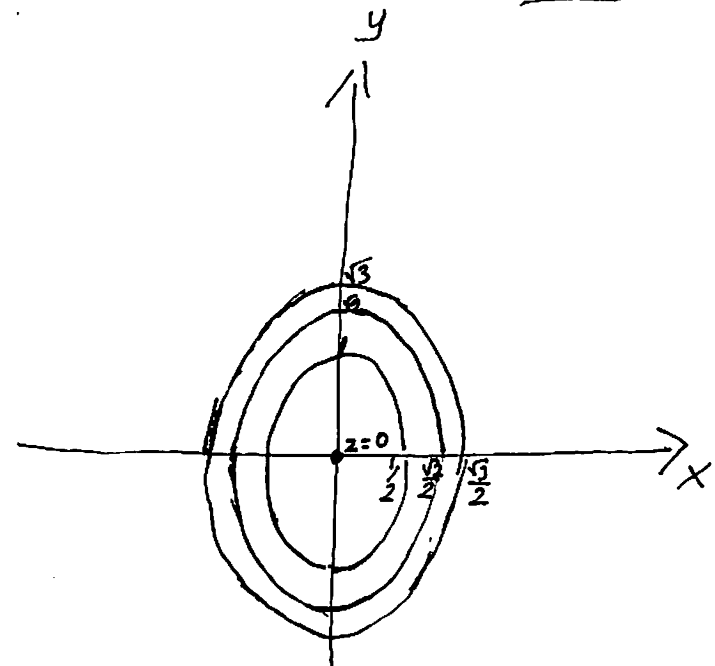
ellipse with radii 3/2 and 3



This is a cone with elliptic horizontal cross-sections.

Therefore the contour-diagram consists of equally spaced ellipses.

$\boxed{\text{The slope is constant everywhere.}}$



b. At $z=0$

$4x^2+y^2=0$

$z=1$

$4x^2+y^2=1$ — ellipse with radii $1/2$ and 1

$z=2$

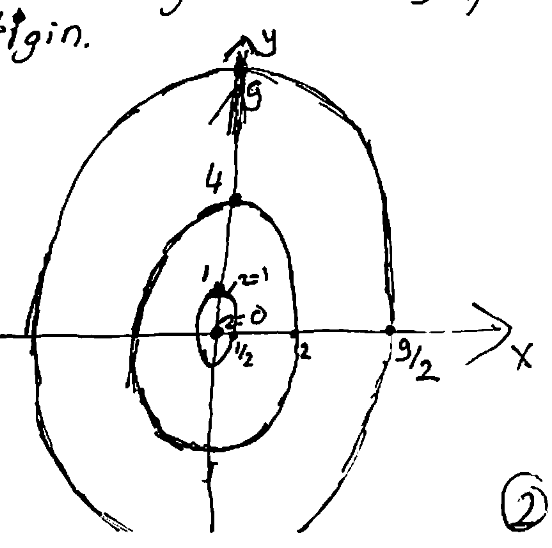
$4x^2+y^2=2$ — ellipse with radii $1/\sqrt{2}$ and $\sqrt{2}$

$z=3$

$4x^2+y^2=3$ — ellipse with radii $\sqrt{3/2}$ and $\sqrt{3}$

$\boxed{\text{The function is becoming steeper as the points are chosen away from the origin}}$

The contour diagram consists of ellipses getting close to each other as you move away from the origin.



c. At $z=0$

$\sqrt[4]{4x^2+y^2}=0$

$z=1$

ellipse with radii $1/2$ and 1 — $\sqrt[4]{4x^2+y^2}=1 \Rightarrow 4x^2+y^2=1$

$z=2$

ellipse with radii — $\sqrt[4]{4x^2+y^2}=2 \Rightarrow 4x^2+y^2=16$

$z=3$

$\sqrt[4]{4x^2+y^2}=3$

ellipse with radii $9/2$ and 9 — $4x^2+y^2=81$

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The contour diagram consists of ellipses getting away from each other as you move away from the origin

$\boxed{\text{The function is steeper towards the origin.}}$

d. At $z=0$ $\quad 0=\sqrt{9-4x^2-y^2}$

ellipse with radii $\frac{3}{2}$ and 3 $\quad 4x^2+y^2=9$

At $z=1$ $\quad 1=\sqrt{9-4x^2-y^2}$

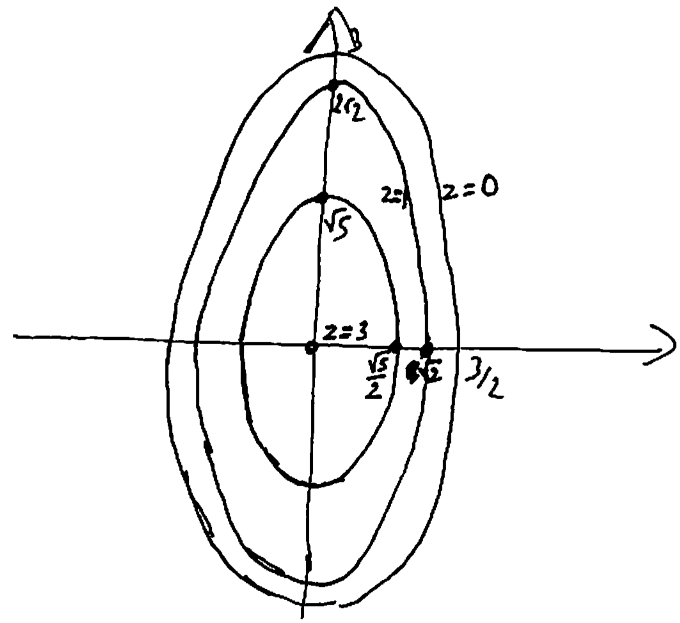
ellipse with radii $\sqrt{2}, 2\sqrt{2}$ $\quad 4x^2+y^2=8$

At $z=2$ $\quad 2=\sqrt{9-4x^2-y^2}$

ellipse with radii $\frac{\sqrt{5}}{2}, \sqrt{5}$ $\quad 4x^2+y^2=5$

At $z=3$ $\quad 3=\sqrt{9-4x^2-y^2}$

~~ellipse with radii~~ point $(0,0)$ $\quad 4x^2+y^2=0$



$z=\sqrt{9-4x^2-y^2}$ is the upper part of the ellipsoid with radii $3/2, 3, 3$.
It has elliptic vertical and horizontal cross-sections

The contour diagram consist of ellipses getting close to each other as you move away from the origin.

$\boxed{\text{The function is becoming steeper as the points are chosen away from the origin}}$

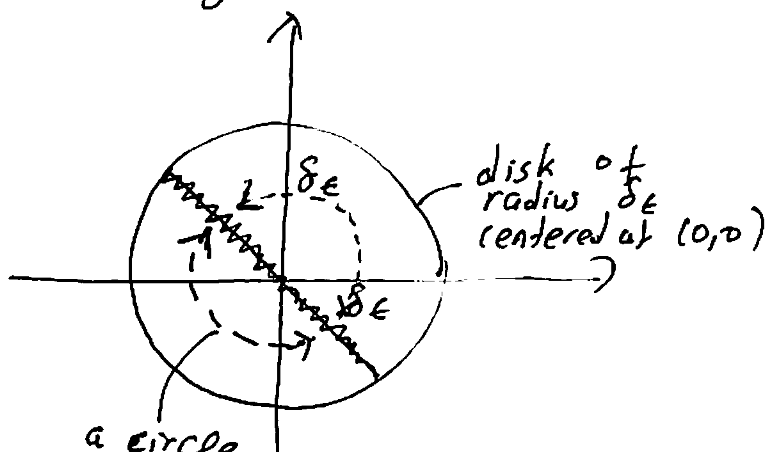
3. a. $$\boxed{\begin{aligned}\lim_{(x,y)\to(\sqrt{\pi/2},\sqrt{\pi/2})} \sin(x^2+xy+y^2) &= \sin\left(\left(\sqrt{\pi/2}\right)^2+\sqrt{\pi/2}\sqrt{\pi/2}+\left(\sqrt{\pi/2}\right)^2\right)\\ &= \sin\left(\tfrac{3\pi}{2}\right)=-1\end{aligned}}$$

since $f(x)=\sin(x)$ and $g(x,y)=x^2+xy+y^2$ are continuous, their composition $h(x,y)=f(g(x,y))$ is also continuous. We used the fact that $h(x,y)$ is continuous, when finding the limit above.

b. $\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2}{x+y} \text{ does not exist}}$ because

$f(x,y)$

$f(x,y)$ is not defined along the line $x=-y$ and as we approach this line the function approaches ∞ or $-\infty$. No matter how small you choose ~~the disk of ra~~ the radius δ_ϵ of a disk centered at $(0,0)$, there will be points inside the disk where function takes values greater than ϵ. (indeed arbitrarily large



a circle inside the disk with radius $r < \delta_\epsilon$.
As you move towards the line $x=-y$, the function either goes to ∞ or $-\infty$. Therefore there are points inside the disk of radius δ_ϵ ~~with values~~ where the function is greater than ϵ.

c. ~~lim~~ Using the squeeze theorem, since

$$-\frac{1}{xy} \leq \frac{\sin(xy)}{(xy)} \leq \frac{1}{xy} \quad \text{for all } (x,y)$$

we obtain

$$\underbrace{\lim_{(x,y)\to(\infty,\infty)} -\frac{1}{xy}}_{0} \leq \lim_{(x,y)\to(\infty,\infty)} \frac{\sin(xy)}{xy} \leq \underbrace{\lim_{(x,y)\to(\infty,\infty)} \frac{1}{xy}}_{0}$$

Therefore

$$\lim_{(x,y)\to(\infty,\infty)} \frac{\sin(xy)}{xy} = 0$$

d. $\boxed{\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{|x|} \text{ does not exist.}}$

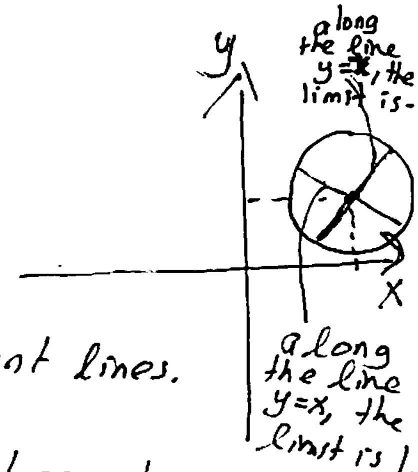
The reasoning is similar to part b. Along the line $x=0$, the function is undefined. Furthermore as we approach the line $x=0$ in the domain along the circles centered at $(x,y)=(0,0)$, the function blows up in absolute value.

4. a. Pick any line that passes through $(x,y)=(2,1)$ in the form

$$y-1=k(x-2).$$

Along the line

$$\frac{x-2}{y-1}=\frac{x-2}{k(x-2)}=\frac{1}{k}$$



Therefore

$$\lim_{(x,y)\to(2,1)} \frac{x-2}{y-1}$$

is different along different lines.

$$\lim_{(x,y)\to(2,1)} f(x,y)=\lim_{(x,y)\to(2,1)} \frac{x-2}{y-1} \text{ does not exist}$$

$\boxed{\text{The function is discontinuous at } (2,1).}$

b. Both of the functions $f(x,y)=x-2$ and $g(x,y)=y$ are continuous everywhere.

$\boxed{\text{Since } g(2,1)=1\neq 0, \text{ the function } u(x,y)=\dfrac{f(x,y)}{g(x,y)}=\dfrac{x-2}{y} \text{ is also continuous at } (x,y)=(2,1).}$

c. The function $g(x,y) = \frac{(x-2)^2}{|y-1|}$ is not defined at $(x,y) = (2,1)$. Therefore it cannot be continuous at $(2,1)$.

d. The limit

$$\lim_{(x,y)\to(2,1)} h(x,y) = \lim_{(x,y)\to(2,1)} \frac{(x-2)^4}{(y-1)^2}$$

does not exist, because of a similar reason as 3.b and 3.d. Along the line $y=1$, the function $h(x,y)$ is undefined, and particularly blows up in absolute value as we approach the line $y=1$.

Therefore $h(x,y)$ is not continuous at $(x,y)=(2,1)$.

5. a. $f_x(x,y) = 3x^2$ $\quad f_y(x,y) = -\sin(y)$ — first derivatives

$f_{xx}(x,y) = 6x$ $\quad f_{yy}(x,y) = -\cos(y)$
$f_{xy}(x,y) = 0$ $\quad f_{yx}(x,y) = 0$ — second derivatives

$$f_{xy}(x,y) = f_{yx}(x,y) = 0$$

b. $f_x(x,y) = 3x^2\cos(y)$ $\quad f_y(x,y) = -x^3\sin(y)$ — first derivatives

$f_{xx}(x,y) = 6x\cos(y)$ $\quad f_{yy}(x,y) = -x^3\cos(y)$
$f_{xy}(x,y) = -3x^2\sin(y)$ $\quad f_{yx}(x,y) = -3x^2\sin y$ — second derivative

$$f_{xy}(x,y) = f_{yx}(x,y) = 0$$

c. First derivatives - By using the chain rule

$f_x(x,y) = -\sin(x^3y)3x^2y$ $\quad f_y(x,y) = -\sin(x^3y)\,x^3$

Second derivatives - By using the chain and product rule

$f_{xx}(x,y) = -\cos(x^3y)9x^4y - \sin(x^3y)6xy$ $\quad f_{yy}(x,y) = -\cos(x^3y)\,x^6$
$f_{xy}(x,y) = -\cos(x^3y)3x^5y - \sin(x^3y)3x^2$ $\quad f_{yx}(x,y) = -\cos(x^3y)\,3x^5y - 3x^2\sin(x^3y)$

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$f_{xy}(x,y) = f_{yx}(x,y) = -3x^5y\cos(x^3y) - 3x^2\sin(x^3y)$

d. First derivatives:

By using the power rule and the chain rule

$$\boxed{f_x(x,y) = 3y(\cos(x))^{3y-1}(-\sin(x))}$$

By using the chain rule

$$\boxed{f_y(x,y) = \frac{\partial((\cos(x))^{3y})}{\partial y} = \frac{\partial(e^{\ln(\cos(x))3y})}{\partial y} = 3\ln(\cos(x))e^{\ln(\cos(x))3y} = 3\ln(\cos(x))(\cos(x))^{3y}}$$

Second derivatives:

$$f_{xx}(x,y) = 3y(3y-1)(\cos(x))^{3y-2}(-\sin(x))^2 - 3y\cos(x)(\cos(x))^{3y-1}$$

$$= \boxed{3y(3y-1)(-\sin(x))^2(\cos(x))^{3y-2} - 3y(\cos(x))^{3y}}$$

$e^{(\ln(\cos(x)))(3y-1)}$

$$f_{xy}(x,y) = 3(\cos(x))^{3y-1}(-\sin(x)) + \frac{\partial((\cos(x))^{3y-1})}{\partial y}3y(-\sin x)$$

$$= 3(\cos(x))^{3y-1}(-\sin(x)) + 3\ln(\cos(x))(\cos(x))^{3y-1}(3y(-\sin(x)))$$

$$= \boxed{3(-\sin(x))(\cos(x))^{3y-1}(1 + \ln(\cos(x))3y)}$$

$$f_{yy}(x,y) = \boxed{(3\ln(\cos(x)))^2(\cos(x))^{3y}}$$

$$f_{yx}(x,y) = \frac{3}{\cos(x)}(-\sin(x))(\cos(x))^{3y} + 3\ln(\cos(x))3y(\cos(x))^{3y-1}(-\sin x)$$

$$= \boxed{3(-\sin(x))(\cos(x))^{3y-1}(1 + 3y\ln(\cos(x)))},$$

$$\boxed{f_{xy}(x,y) = f_{yx}(x,y) = 3(-\sin(x))(\cos(x))^{3y-1}(1+3y\ln(\cos(x)))}$$

7. a The equation of the tangent plane

$$z = f(1,e) + f_x(1,e)(x-1) + f_y(1,e)(y-e)$$

where

$$f(1,e) = 1 \ln(e) = 1$$

$$f_x(1,e) = \ln(1 \cdot e) + 1 = 2$$

$$f_y(1,e) = 1/e$$

Note that $f_x(x,y) = \ln(xy) + \frac{xy}{xy} = \ln(xy) + 1$

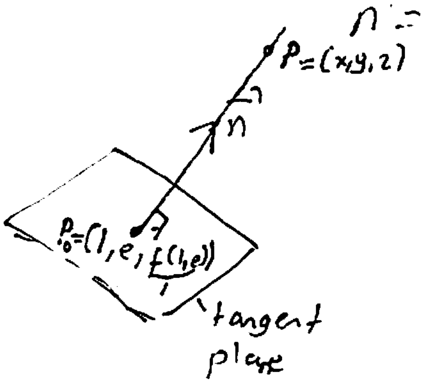
$f_y(x,y) = \frac{x\,x}{x\,y} = \frac{x}{y}$

Therefore

$$\boxed{z = 1 + 2(x-1) + 1/e\,(y-e)}$$

b. The normal to the tangent plane at $(1,e)$ is

$$\vec{n} = (f_x(1,e), f_y(1,e), -1) = (2, 1/e, -1)$$



Pick a point $P=(x,y,z)$ on the normal line

$\vec{P_0P} = t\vec{n}$ (since $\vec{P_0P}$ and $\vec{n}$ are parallel or opposite to each other)

t: scalar

$$(x-1, y-e, z-1) = t(2, \tfrac{1}{e}, -1)$$

Equation of the normal line: $\boxed{x = 1+2t,\ y = e + \frac{t}{e},\ z = 1-t}$

c. Since the tangent line lies on $x=1$, its slope is given by $f_y(1,e) = 1/e$. It also passes through $(1, e, f(1,e))$

The symmetric equation for the line

$x = 1$ (since the line is on the plane $x=1$) $\quad z - f(1,e) = f_y(1,e)(y-e)$

$$z - 1 = 1/e\,(y-e) \implies z = \frac{y}{e} + 1 -$$

Symmetric equation for the tangent line on the plane $x=1$: $\boxed{x = 1,\ z = \frac{1}{e}y}$

d. Since y is fixed at e, the slope of the line is given by $f_x(1,e) = 2$ and it also passes through $(1, e, f(1,e))$.

$$y = e,\quad z - f(1,e) = f_x(1,e)(x-1)$$

$$z - 1 = 2(x-1) \implies z = 2x - 1$$

Symmetric equation for the tangent line on the plane $y=e$: $\boxed{y = e,\ z = 2x-1}$

8: a. The linear approximation around (20, 60) is given by

$$(\star)\quad L(R_1, R_2) = R(20,60) + R_{R_1}(20,60)(R_1-20) + R_{R_2}(20,60)(R_2-60)$$

Using implicit differentiation

$$\frac{\partial(1/R)}{\partial R_1} = \frac{\partial\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}{\partial R_1} \Rightarrow \frac{+1}{(R(R_1,R_2))^2} R_{R_1}(R_1,R_2) = +\frac{1}{R_1^2}$$

$$\Rightarrow R_{R_1}(R_1,R_2) = \frac{(R(R_1,R_2))^2}{R_1^2} \Rightarrow R_{R_1}(20,60) = \frac{(R(20,60))^2}{(20)^2}$$

Similarly

$$R_{R_2}(R_1,R_2) = \frac{(R(R_1,R_2))^2}{R_2^2} \Rightarrow R_{R_2}(20,60) = \frac{(R(20,60))^2}{(60)^2}$$

Noting that $\frac{1}{R(20,60)} = \frac{1}{20} + \frac{1}{60}$ or $\underline{R(20,60) = 15}$, we have

$$R_{R_1}(20,60) = \frac{(15)^2}{(20)^2} = \frac{9}{16} \quad \text{and } R_{R_2}(20,60) = \frac{(15)^2}{(60)^2} = \frac{1}{16}.$$

Plugging in $R(20,60) = 15$, $R_{R_1}(20,60) = \frac{9}{16}$ and $R_{R_2}(20,60) = \frac{1}{16}$ in the equation $(\star)$, we obtain

$$L(R_1, R_2) = 15 + \frac{9}{16}(R_1-20) + \frac{1}{16}(R_2-60)$$

b. We want to estimate $R(R_1, R_2)$ at $R_1 = 20 + (0.01)20 = 20.2$ and $R_2 = 60 + (0.01)60 = 60.6$ using the linear approximation

$$\boxed{\begin{aligned} L(20.2, 60.6) &= 15 + \frac{9}{16}(0.2) + \frac{1}{16}(0.6) \\ &= 15 + \overset{0.15}{\frac{2.4}{16}} = 15.15 \end{aligned}}$$

is the value, approximately of R at $(20.2, 60.6)$.

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