Math 20C (Lecture C) - Homework 4

(Due on November 28th, Wednesday by 3pm)

This homework covers sections 14.5-8 from Stewart's book. You need to return only the questions marked with (∗). The others are provided for practicing purposes.

1.(*) (Section 14.5) On a spring day you decide to hike on a hill close to your home. You follow a route given by a parametric equation $(x(t), y(t), z(t))$ with t denoting the time and $(x(t), y(t), z(t))$ denoting your position at time t. The density ρ of the bottle of water which you carry with you varies as a function of the temperature T and pressure P, that is $\rho = f(P, T)$. The temperature and pressure vary with respect to your position, that is $T = g(x(t), y(t), z(t))$ and $P = h(x(t), y(t), z(t))$. Express the rate of change of the density of your water with respect to time $\frac{d\rho}{dt}$ in terms of the derivatives $\frac{\partial \rho}{\partial P}$, $\frac{\partial \rho}{\partial T}$, $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial P}{\partial z}$, $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$, $x'(t)$, $y'(t)$ and $z'(t)$.

2.(*) (Section 14.5) In each part use the chain rule to find the indicated derivatives of the functions.

a. Let $h(t) = f(x(t), y(t)) = e^{x+y}$ where $x(t) = \frac{1}{t}$ and $y(t) = t^2$. Find $h'(t)$.

b. Let $h(t) = f(x(t), y(t), t) = te^{x+y}$ where $x = \frac{1}{t}$ and $y = t^2$. Find $h'(t)$.

c. Let $h(x, y) = f(u(y), v(x)) = uv$ where $u(y) = y^2$ and $v(x) = x^2$. Find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

d. Let $h(x, y) = f(u(x), v(x, y)) = \frac{u}{v}$ where $u(x) = \cos(x)$ and $v(x, y) = e^y \sin(x)$. Find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

3.(*) (Section 14.6) Let

$$f(x, y) = x^2 y + y^2 x$$

a. Find the gradient vector $\nabla f(1, 2)$.

b. Find the directional derivative $D_{\vec{u}} f(1, 2)$ in the direction $\vec{u} = \frac{4}{5}\vec{\imath} + \frac{3}{5}\vec{\jmath}$.

c. Find the direction $\vec{u}$ so that $D_{\vec{u}} f(1, 2)$ is as large as possible.

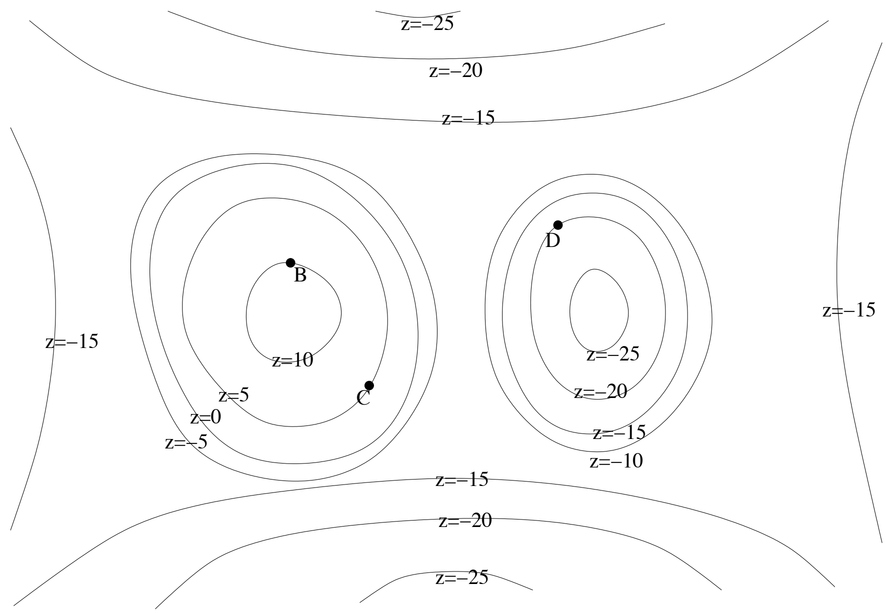
d. Find the direction $\vec{u}$ so that $D_{\vec{u}} f(1, 2) = 0$.

4. (Section 14.6) Show that the plane tangent to $z = f(x, y)$ at $(x, y) = (a, b)$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

contains all of the tangent lines to f at $(x, y) = (a, b)$. (Hint: Find the angle between a vector in the direction of a given tangent line and the normal vector to the tangent plane.)

5.(*) (Section 14.1-6-7) Answer all of the parts for the function whose contour diagram is illustrated below.



a. Among the points B, C and D determine where the function is steepest. Explain your reasoning.

b. Draw vectors on the contour diagram at C and D pointing in the direction of the gradient vectors. Explain in words also how the gradients vectors are aligned with respect to the contours.

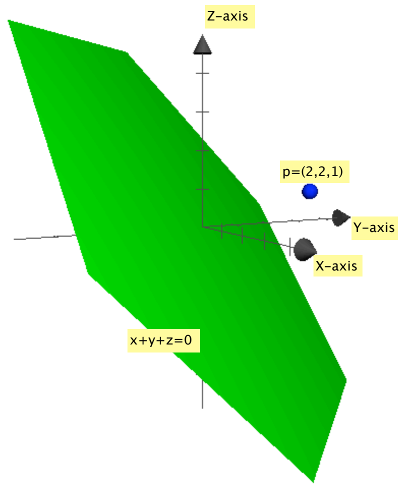
c. Mark three points on the contour diagram that are possibly a local minimum, a local maximum and a saddle point with the letters M, X and S, respectively.

6. (Section 14.7) Find the critical points of the functions below. In each part determine also whether a critical point is a local minimum, local maximum or saddle point.

a. $f(x, y) = (x - 3)^2 - (y - 1)^2$ **b.** $f(x, y) = e^{-(x-3)^2-(y-1)^2}$

c. $f(x, y) = -x^2 - xy - xy^2$ **d.** $f(x, y) = e^{x^2} \cos(y)$

7.(*) (Section 14.7) Find the unique point on the plane $x + y + z = 0$ that is closest to the point $p = (2, 2, 1)$ in 3-space by setting up an (unconstrained) optimization problem depending on two variables. The function that needs to be minimized is the distance from the point p to a point p_1 on the plane. The closest point should be a critical point of the optimization problem.



8. (Section 14.7) You measure the displacement $x(t)$ of an object at times $t = 1, 2, 3$ relative to its location at time $t = 0$ (that is $x(0) = 0$) as listed in the table

t	1	2	3
$x(t)$	10	20	40

The object has a constant acceleration a and an unknown initial velocity v. The displacement of the object is given by the formula

$$x(t) = \frac{a}{2}t^2 + vt.$$

Based on your measurements your task is to determine the acceleration a and the initial velocity v that minimize the sum of the square of errors at time $t = 1, 2, 3$,

$$(\text{Error at } t = 1)^2 + (\text{Error at } t = 2)^2 + (\text{Error at } t = 3)^2 = (x(1)-10)^2+(x(2)-20)^2+(x(3)-40)^2.$$

9.(*) (Section 14.8) Consider a box without a lid whose base is a square and volume is $4000cm^3$. Find the dimensions of such a box with minimum surface area by posing it as a constrained optimization problem and using Lagrange multipliers