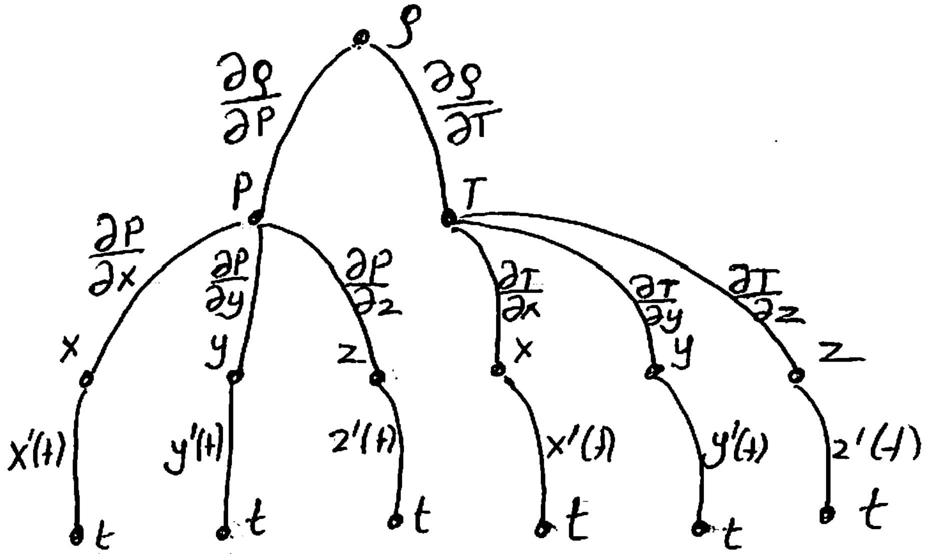
Homework 4 - Answers to Selected Questions

1. We use a tree-like diagram. If z is a function of x, we connect them by an edge and label it with $\frac{\partial z}{\partial x}$ or $\frac{dz}{dx}$ (depending on whether z depends only on x). Each path from ρ to t yields the product of the derivatives along the way. The derivative $\frac{d\rho}{dt}$ is the sum of expressions produced by various paths from ρ to t.



$$\frac{d\rho}{dt} = \underbrace{\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial x}x'(t)}_{\text{Path through } P \text{ and } x} + \underbrace{\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial y}y'(t)}_{\text{Path through } P \text{ and } y} + \underbrace{\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial z}z'(t)}_{\text{Path through } P \text{ and } z}$$

$$+ \underbrace{\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial x}x'(t)}_{\text{Path through } T \text{ and } x} + \underbrace{\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial y}y'(t)}_{\text{Path through } T \text{ and } y} + \underbrace{\frac{\partial \rho}{\partial T}\frac{\partial T}{\partial z}z'(t)}_{\text{Path through } T \text{ and } z}$$

2.

a. $h'(t) = f_1(x(t), y(t))\, x'(t) + f_2(x(t), y(t))\, y'(t)$

where

$$f_1(x,y) = e^{x+y} \qquad f_2(x,y) = e^{x+y}$$

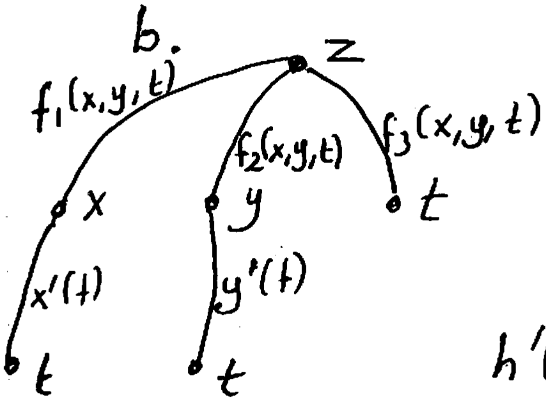
$$x'(t) = -1/t^2 \qquad y'(t) = 2t$$

Therefore

$$h'(t) = e^{x+y}\left(-1/t^2\right) + e^{x+y}(2t)$$

$$= e^{x+y}\left(2t - 1/t^2\right) = \boxed{e^{\left(1/t + t^2\right)}\left(2t - 1/t^2\right)}$$

b.



Use the tree-like diagram again

Let $z = h(t)$.

$$h'(t) = \frac{dz}{dt} = f_1(x,y,t)\, x'(t) + f_2(x,y,t)\, y'(t) + f_3(x,y,t)$$

where

$$f_1(x,y,t) = te^{x+y}, \quad f_2(x,y,t) = te^{x+y}, \quad f_3(x,y,t) = e^{x+y}$$

$$x'(t) = -1/t^2, \quad y'(t) = 2t$$

Therefore

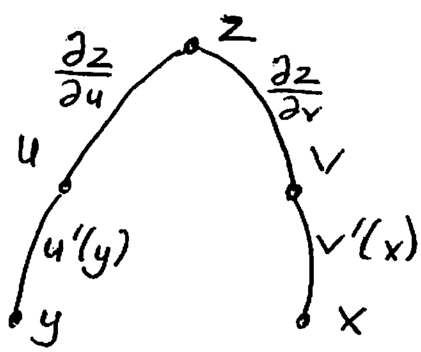
$$h'(t) = t e^{x+y}\left(-1/t^2\right) + t e^{x+y}(2t) + e^{x+y}$$

$$= e^{x+y}\left(2t^2 - 1/t + 1\right) = \boxed{e^{\left(\frac{1}{t} + t^2\right)}\left(2t^2 - 1/t + 1\right)}$$

c.

$$\cancel{\frac{\partial h}{\partial x} = f_1(u(y), v(x))}$$

Use a tree-like diagram. Let $z = h(x,y)$



$$\frac{\partial h}{\partial y} = \left.\frac{\partial z}{\partial u}\right|_{\substack{u=u(y)\\ v=v(x)}} u'(y)$$

$$= v(x)\, 2y$$

$$= \boxed{x^2 2y}$$

Above we used the derivatives $\frac{\partial z}{\partial u} = v$ and $u'(y) = 2y$

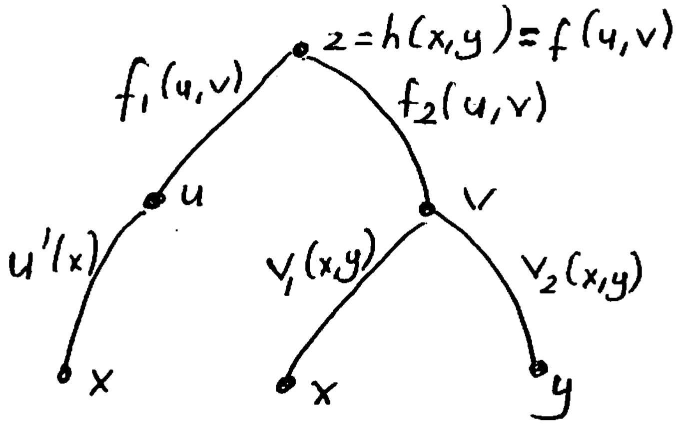
$$\frac{\partial h}{\partial x} = \left.\frac{\partial z}{\partial v}\right|_{\substack{u=u(y)\\ v=v(x)}} v'(x)$$

$$= u(y)\, 2x$$

$$= \boxed{y^2 2x}$$

Above we used $\frac{\partial z}{\partial v} = u$ and $v'(x) = 2x$

d. Using the tree-like diagram



We need to calculate the derivatives of f, u and v.

$$f_1(u,v) = 1/v$$
$$f_2(u,v) = -u/v^2$$
$$u'(x) = -\sin(x)$$
$$v_1(x,y) = e^y \cos(x)$$
$$v_2(x,y) = e^y \sin(x)$$

$$\frac{\partial h}{\partial x} = f_1(u,v)\,u'(x) + f_2(u,v)\,v_1(x,y)$$
$$= 1/v\,(-\sin(x)) + \left(-\frac{u}{v^2}\right) e^y \cos(x)$$
$$= 1/x^2\,(-\sin(x)) + \left(\frac{-\cos(x)}{e^{2y}\sin^2(x)}\right) e^y \cos(x)$$

$$= \boxed{\frac{-\sin(x)}{x^2} - \frac{\cot an^2(x)}{e^y}}$$

(Recall that $\text{cotan}(x) = \frac{\cos(x)}{\sin(x)}$)

$$\frac{\partial h}{\partial y} = f_2(u,v)\,v_2(x,y) = -u/v^2\, e^y \sin(x)$$
$$= -\frac{\cos(x)}{e^{2y}\sin^2(x)}\, e^y \sin(x) = \boxed{\frac{-\text{cotan}(x)}{e^y}}$$

3.

a.

$$\nabla f(1,2) = \left(\left.\frac{\partial f}{\partial x}\right|_{\substack{x=1\\y=2}}, \left.\frac{\partial f}{\partial y}\right|_{\substack{x=1\\y=2}} \right)$$

$$= \left(\left. 2xy + y^2 \right|_{\substack{x=1\\y=2}}, \left. x^2 + 2yx \right|_{\substack{x=1\\y=2}} \right)$$

$$= \boxed{(8, 5)}$$

b.

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

$$= (8,5) \cdot \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$= \boxed{\frac{47}{5}}$$

c. $D_{\vec{u}} f(1,2)$ is maximized if $\vec{u}$ points in the direction of $\nabla f(1,2)$.

$\boxed{\vec{u} = (8,5)}$ is a possible right answer

or you can normalize (make it a unit vector)

$$\boxed{\vec{u} = \frac{1}{\sqrt{64+25}}(8,5) = \frac{1}{\sqrt{89}}(8,5)}$$

d. $D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$

The dot product is 0, if

$$\nabla f(1,2) \perp \vec{u}$$

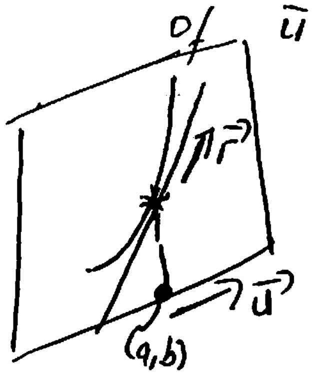
Switching the order of the components and negating the sign of one of the components of $\nabla f(1,2)$, yield a vector perpendicular to $\nabla f(1,2)$

$\boxed{\vec{u} = (-5, 8)}$ is a possible right anwer.

or you can normalize it.

$$\boxed{\vec{u} = \frac{1}{\sqrt{89}}(-5, 8)}$$

4. Consider line tangent at $(x,y)=(a,b)$ contained on the plane in the direction of $\vec{u}$?



To stay on the line we have to move in the direction

$$\vec{r} = (u_1, u_2, D_{\vec{u}} f(a,b))$$

parallel to the tangent line

where $\vec{u} = (u_1, u_2)$

Normal to a plane

$$m_x x + m_y y + m_z z = c$$

is given by the coefficients of x, y, z
(m_x, m_y, m_z)

In particular normal to the tangent plane

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

or reorganizing this equation

$$\underbrace{-f(a,b) + f_x(a,b)a + f_y(a,b)b}_{\text{constant}} = f_x(a,b)x + f_y(a,b)y - z$$

we see that the normal to the tangent plane is

$$\vec{n} = (f_x(a,b),\ f_y(a,b), -1)$$

If we take the dot product of the normal vector $\vec{n}$ and the direction vector for the tangent line, we obtain

$$\vec{n} \cdot \vec{r} = (f_x(a,b), f_y(a,b), -1) \cdot (u_1, u_2, D_{\vec{u}} f(a,b))$$

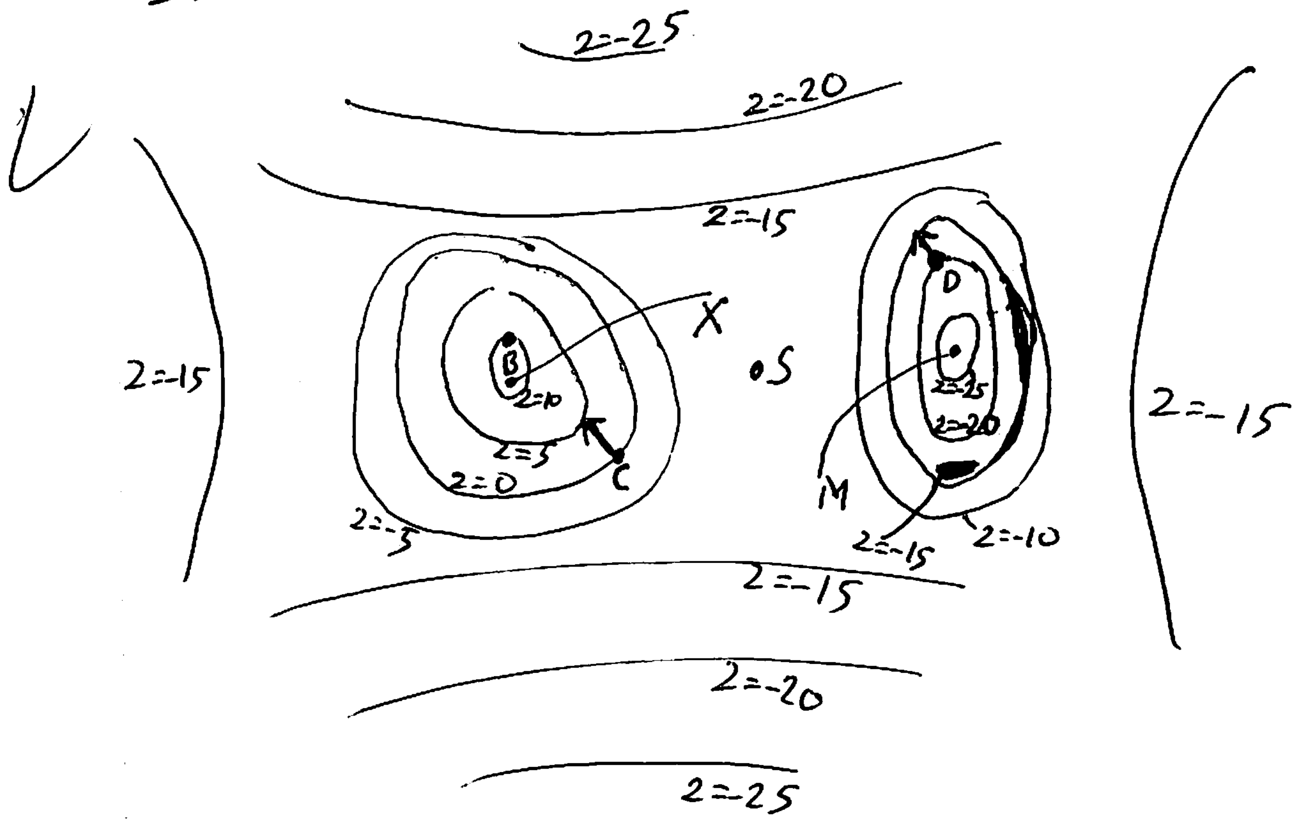
$$= f_x(a,b)u_1 + f_y(a,b)u_2 - D_{\vec{u}} f(a,b)$$

$$= \nabla f \cdot \vec{u} - D_{\vec{u}} f(a,b)$$

$$= D_{\vec{u}} f(a,b) - D_{\vec{u}} f(a,b) = 0.$$

Therefore $\vec{n}$ is perpendicular to $\vec{r}$, that is the tangent line is contained on the tangent plane.

5.



a. The function is steepest at D. Because the spacing between the contour diagrams is smallest at D.

b. The gradient vectors are illustrated on the contour diagrams. At any point the gradient vector is perpendicular to the tangent line at that point, and points towards the level set with a greater z value.

c. Illustrated on the contour diagram. At the indicated saddle point it seems the function has an ~~[illegible]~~ inflection point in one direction and a local maximum in the other direction.

6.

a. $\nabla f(x,y) = (2(x-3), -2(y-11))$

$\boxed{(x,y) = (3,11)}$ is a critical point

This is a saddle point, because the function has a local minimum at this point when y is fixed and a local maximum when x is fixed.

b. $\nabla f(x,y) = \left(-2(x-3)\, e^{-(x-3)^2-(y-1)^2}, -2(y-1)\, e^{-(x-3)^2-(y-1)^2}\right)$

$\boxed{(x,y) = (3,1)}$ is the only critical point.

This is a local and global maximum, because the exponent is always non-positive, meaning the function is less than or equal to one at all (x,y). It is exactly equal to one at $(3,1)$.

c. $\nabla f(x,y) = (-2x - y - y^2, -x - 2xy)$

$= (0,0)$

$\Longrightarrow$ (i) $x = -2xy$ (ii) $y^2 + y = -2x$

Clearly $\boxed{(0,0)}$ is a critical point

Assuming $x \neq 0$, ~~from (i)~~

from (i) $y = -\frac{1}{2}$ and from (ii) $\frac{1}{4} + \left(-\frac{1}{2}\right) = -2x$

$x = \frac{1}{8}$

so $\boxed{\left(\frac{1}{8}, -\frac{1}{2}\right)}$ is the other critical point.

Use the second derivative test

$$D = \begin{bmatrix} f_{xx}(0,0), & f_{xy}(0,0) \\ f_{yx}(0,0), & f_{yy}(0,0) \end{bmatrix} \quad \text{where } \begin{aligned} f_{xx}(0,0) &= -2 \\ f_{xy}(0,0) = f_{yx}(0,0) &= -1 \\ f_{yy}(0,0) &= 0 \end{aligned}$$

$$= \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

determinant of D is $(-2)(0)-(-1)(-1) = -1 < 0$.
Therefore $(0,0)$ is a saddle point.

$$D = \begin{bmatrix} f_{xx}(\tfrac{1}{8},-\tfrac{1}{2}) & f_{xy}(\tfrac{1}{8},-\tfrac{1}{2}) \\ f_{yx}(\tfrac{1}{8},-\tfrac{1}{2}) & f_{yy}(\tfrac{1}{8},-\tfrac{1}{2}) \end{bmatrix} \quad \text{where } \begin{aligned} f_{xx}(\tfrac{1}{8},-\tfrac{1}{2}) &= -2 \\ f_{xy}(\tfrac{1}{8},-\tfrac{1}{2}) = f_{yx}(\tfrac{1}{8},-\tfrac{1}{2}) &= 0 \\ f_{yy}(\tfrac{1}{8},-\tfrac{1}{2}) &= -\tfrac{1}{4} \end{aligned}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1/4 \end{bmatrix}$$

$\det(D) = -2\left(-\frac{1}{4}\right) = \frac{1}{2} > 0$

and

$f_{xx}(\frac{1}{8},-\frac{1}{2}) < 0$

$(\frac{1}{8},-\frac{1}{2})$ is a local maximum d.

$$\nabla f(x,y) = (2xe^{x^2}\cos(y), -e^{x^2}\sin(y))$$
$$= (0, 0)$$

$$e^{-x^2}\sin(y) = 0 \implies y = 0 + [redacted]\pi k \text{ for any integer } k$$
$$2xe^{x^2}\cos(y) = 0 \implies x = 0$$

There are Infinitely many critical points

$$\boxed{(0, 2\pi k) \text{ for all integer } k}$$

~~$e^{x^2}\cos(y)$ when y is fixed, takes the smallest value at $x=0$~~

~~$e^{x^2}\cos(y)$ when x is fixed tak~~

Hessian matrix (matrix of second derivatives)

$$D(x,y) = \begin{bmatrix} \underbrace{2e^{x^2}\cos(y) + 4x^2e^{x^2}\cos(y)}_{f_{xx}} & \underbrace{-2xe^{-x^2}\sin(y)}_{f_{xy}} \\ \underbrace{-2xe^{x^2}\sin(y)}_{f_{yx}} & \underbrace{-e^{-x^2}\cos(y)}_{f_{yy}} \end{bmatrix}$$

at $(x,y) = (0, \pi + 2\pi k)$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(D) = -2 < 0$$

$(0, \pi + 2\pi k)$ are saddle points

at $(x,y) = (0, 2\pi k)$, we also have saddle points.

To see this $e^{x^2}\cos(y)$ is minimized at $x=0$ when y is fixed, but maximized at $y = 2\pi k$ when x is fixed.

7. The problem seems like a constrained optimization problem in 3D space,

$$\text{MINIMIZE } \underbrace{\sqrt{(x-2)^2+(y-2)^2+(z-1)^2}}_{d(x,y,z)\text{: distance from a point on the plane to } (2,2,1)}$$

$$\underbrace{x+y+z=0}_{\text{subject to the constraint that the point is on the plane}}$$

But we can eliminate z in the distance function using the constraint $z=-x-y$. Plugging this value in the objective function, we obtain an unconstrained optimization problem.

$$\text{MINIMIZE } \underbrace{\sqrt{(x-2)^2+(y-2)^2+(-x-y-1)^2}}_{f(x,y)}$$

Instead of minimizing $f(x,y)$ we can minimize $f^2(x,y)$, since for the global minimum

$$f(x_*,y_*) \leqslant f(x,y) \quad \text{for all } x \text{ and } y$$

holds, which is equivalent to

$$f^2(x_*,y_*) \leqslant f^2(x,y) \quad \text{for all } x \text{ and } y.$$

Therefore f and f_a share the same global minimum. Therefore we need to solve the problem

$$\text{MINIMIZE} \quad \underbrace{(x-2)^2+(y-2)^2+(x+y+1)^2}_{h(x,y)}$$

$$\nabla h(x,y) = \big(2(x-2)+2(x+y+1),\ 2(y-2)+2(x+y+1)\big)$$
$$= (0,0)$$

We obtain two equations in two unknowns

$$\cancel{2}(x-2)+\cancel{2}(x+y+1)=0 \qquad \cancel{2}(y-2)+\cancel{2}(x+y+1)=0$$
$$(i)\ 2x+y=1 \qquad\qquad (ii)\ x+2y=1$$

Multiply the second one by +2 and subtract it from the first to obtain

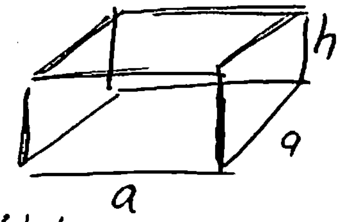
$$-3y=-1 \Rightarrow y=\frac{1}{3}$$

Plug $y=\frac{1}{3}$ in (i)

$$x=\frac{1}{3}$$

The closest point is $\left(\frac{1}{3},\frac{1}{3}\right)$

9.



Unknowns
a: base length
h: height

We want to minimize the surface area
$S(r,h) = a^2 + 4ah$
subject to the constraint that the volume is 4000 cm³, that is
$V(a,h) = ha^2 = 4000$

Constrained problem:	MINIMIZE $S(a,h)$
	$V(a,h) = 4000$

The solution must satisfy the conditions
$$\nabla S(a,h) = \lambda \nabla V(a,h)$$
$$V(a,h) = 4000$$
where
$$\nabla S(a,h) = (2a+4h,\ 4a)$$
$$\nabla V(a,h) = (2ah,\ a^2)$$

We obtain three equations in three unknowns a, h, λ

(i) $\cancel{2}a + \overset{2}{\cancel{4}}h = \lambda \cancel{2}ah$

(ii) $4a = \lambda a^2$

(iii) $ha^2 = 4000$

From (i) $\lambda = (a+2h)/ah$ and (ii) $\lambda = 4/a$

$$\frac{a+2h}{\cancel{a}h} = \frac{4}{\cancel{a}} \Longrightarrow a = 2h$$

(a cannot be zero, if a=0 the constraint is not satisfied)

Plug $a=2h$ in (iii)

$4h^3 = 4000 \Longrightarrow$ $\boxed{h = 10,\ a = 20 \text{ yields the minimal surface area}}$