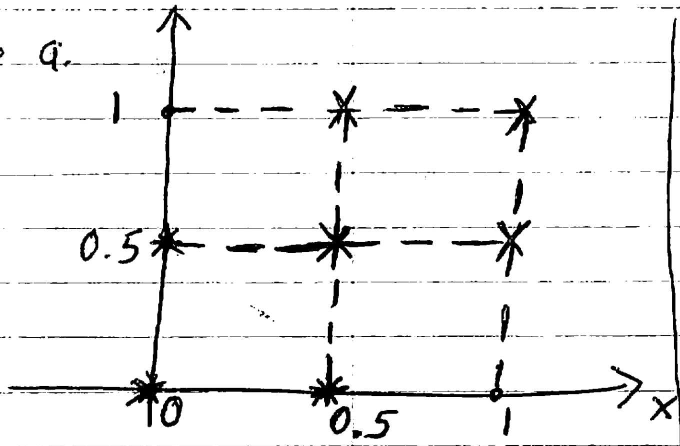
MATH 20C - Homework 5

1. a.



Right sum

The function values are evaluated at the upper right corner.

For $f(x,y) = \sqrt{4-x^2-y^2}$

Right sum $= \Delta x \Delta y \,(f(0.5,0.5) + f(0.5,1) + f(1,0.5) + f(1,1))$

$= 0.25(\sqrt{3.5} + \sqrt{2.75} + \sqrt{2.75} + \sqrt{2})$

$= 1.65$

Left sum

The function values are evaluated at the lower left corner.

For $f(x,y) = \sqrt{4-x^2-y^2}$

Left sum $= \Delta x \Delta y \,(f(0,0) + f(0,0.5) + f(0.5,0) + f(0.5,0.5))$

$= 0.25(2 + \sqrt{3.75} + \sqrt{3.75} + \sqrt{3.5})$

$= 1.94$

For $g(x,y) = \sqrt{x^2+y^2}$

Left sum $= \Delta x \Delta y \,(g(0,0) + g(0,0.5) + g(0.5,0) + g(0.5,0.5))$

$= 0.25(0 + \sqrt{0.25} + \sqrt{0.25} + \sqrt{0.5})$

$= 0.25(2 \times 0.5 + \sqrt{0.5})$

$= 0.43$

For $g(x,y) = \sqrt{x^2+y^2}$

Right sum $= \Delta x \Delta y \,(g(0.5,0.5) + g(0.5,1) + g(1,0.5) + g(1,1))$

$= 0.25(\sqrt{0.5} + \sqrt{1.25} + \sqrt{1.25} + \sqrt{2})$

$= 0.25(\sqrt{0.5} + 2\sqrt{1.25} + \sqrt{2})$

$= 1.09$

b. $f(x,y)$ is decreasing both with respect to x and with respect to y on the indicated domain D.

~~Each box in the left sum lies above the graph of the function~~

~~Therefore for $f(x,y)$ LEFT SUM $\geq$~~

1

$f(x,y)$ takes the largest value at the lower left corner and the smallest value at the upper right corner.
Therefore
For $f(x,y)$
LEFT SUM $\geqslant$ MID-POINT SUM $\geqslant$ RIGHT SUM

$g(x,y)$ is increasing both with respect to x and with respect to y.

$g(x,y)$ takes the largest value in each rectangle at the upper right corner and the smallest value at the lower left corner.
For $g(x,y)$
LEFT SUM $\leqslant$ MID-POINT SUM $\leqslant$ RIGHT SUM

2. a.

$$\iint_D \frac{y}{x}\,dx\,dy = \int_0^1\int_1^2 \frac{y}{x}\,dx\,dy$$

$$= \int_0^1 y\ln(x)\Big|_{x=1}^{2} dy = \int_0^1 y\,(\ln(2))\,dy$$

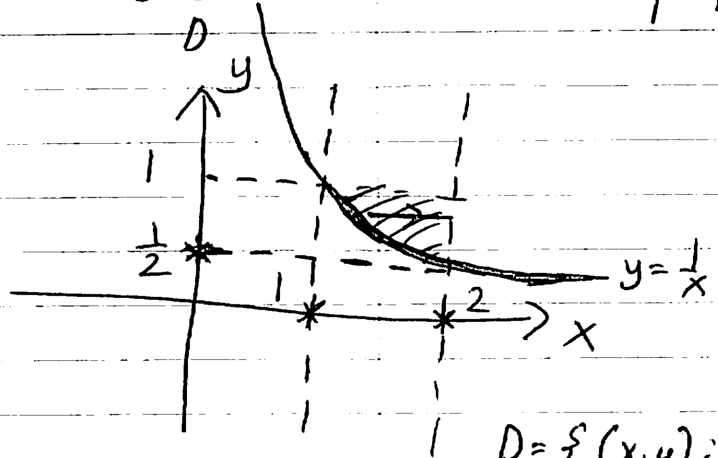
$$= \frac{y^2}{2}\ln(2)\Big|_{y=0}^{1} = \frac{\ln(2)}{2}$$

b.

$$\iint_D y e^{xy}\,dx\,dy = \int_{-1}^{1}\int_{0}^{1} y e^{xy}\,dx\,dy = \int_{-1}^{1} e^{xy}\Big|_{x=0}^{1} dy$$

$$= \int_{-1}^{1} (e^{y} - 1)\,dy = e^{y} - y \Big|_{y=-1}^{1} = (e-1) - (e^{-1}+1)$$
$$= e - \frac{1}{e} - 2$$

c. $$\iint_D y e^{xy}\,dx\,dy = \int_{1}^{2}\int_{1/x}^{1} y e^{xy}\,dy\,dx$$



If we integrate with respect to x first, we wouldn't have to use integration by parts.

$D = \{(x,y) : 1 \leqslant x \leqslant 2,\ \frac{1}{x} \leqslant y \leqslant 1\}$
y-simple

$D = \{(x,y) : \frac{1}{2} \leqslant y \leqslant 1,\ \frac{1}{y} \leqslant x \leqslant 2\}$
x-simple

$$\int_{1}^{2}\int_{1/x}^{1} y e^{xy}\,dy\,dx = \int_{1/2}^{1}\int_{1/y}^{2} y e^{xy}\,dx\,dy$$

$$= \int_{1/2}^{1} e^{xy}\Big|_{x=\frac{1}{y}}^{2} dy = \int_{1/2}^{1} (e^{2y} - e)\,dy$$

$$= \frac{e^{2y}}{2} - ey \Big|_{y=1/2}^{1} = \left(\frac{e^2}{2} - e\right) - \left(\frac{e}{2} - \frac{e}{2}\right) = \frac{e^2}{2} - e$$

d.

$$\iint_D \frac{e^{yx}}{x}\,dx\,dy = \int_1^2\int_{1/x}^{2/x^2} \frac{e^{yx}}{x}\,dy\,dx$$

$$= \int_1^2 \frac{e^{yx}}{x^2}\Big|_{y=1/x}^{2/x^2}\,dx$$

$$= \int_1^2 \left(\frac{e^{2/x}}{x^2} - \frac{e}{x^2}\right)dx = \frac{e^{2/x}}{-2} + \frac{e}{x}\Big|_{x=1}^{2}$$

$$= \left(\frac{e}{-2} + \frac{e}{2}\right) - \left(\frac{e^2}{-2} + e\right)$$

$$= \frac{e^2}{2} - e$$

3. You need to plot the domains in both parts.

a. $D = \{(x,y) : 0 \leq y \leq 2, \frac{-y}{2} \leq x \leq \frac{y}{2}\}$

x-simple

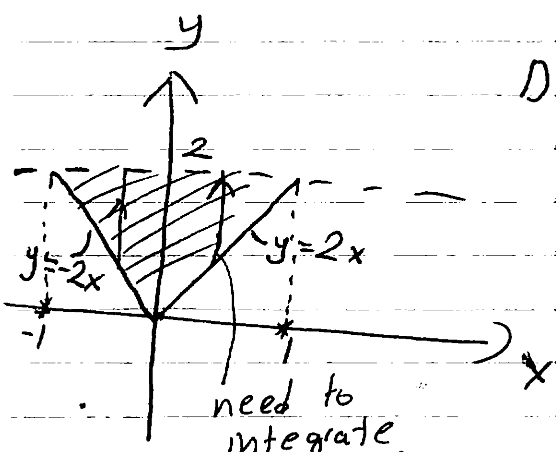
$D = \{(x,y) : 0 \leq x \leq 1, 2x \leq y \leq 2\}$

$\cup$

$\{(x,y) : -1 \leq x \leq 0, -2x \leq y \leq 2\}$

y-simple

(You have to split the integral into two because the lower bounds on the first and second quadrants on y are different)



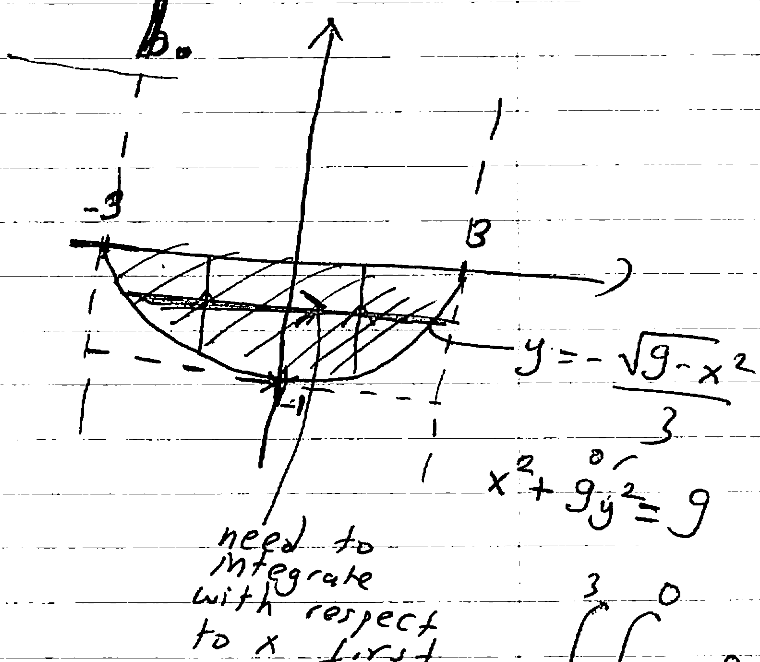
need to integrate vertically first (with respect to y.)

4

$$\int_0^2 \int_{-y/2}^{y/2} f(x,y)\,dx\,dy = \int_0^1 \int_{2x}^{2} f(x,y)\,dy\,dx$$

$$+ \int_{-1}^0 \int_{-2x}^{2} f(x,y)\,dy\,dx$$

b.



$D = \{(x,y) : -3 \leq x \leq 3, -\frac{\sqrt{9-x^2}}{3} \leq y \leq 0\}$

y-simple

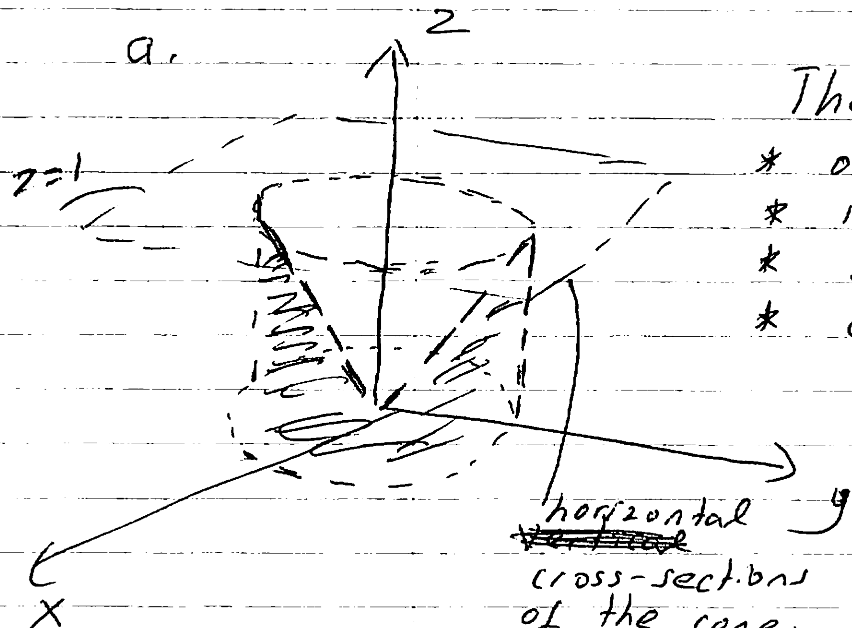
$D = \{(x,y) : -1 \leq x \leq 1, -\frac{\sqrt{9-x^2}}{3} \leq y \leq +\frac{\sqrt{9-x^2}}{3}\}$

x-simple

$$\int_{-3}^{3} \int_{-\frac{\sqrt{9-x^2}}{3}}^{0} f(x,y)\,dy\,dx = \int_{-1}^{1} \int_{-\frac{\sqrt{9-x^2}}{3}}^{\frac{\sqrt{9-x^2}}{3}} f(x,y)\,dx\,dy$$

4. In this type of question you need to visualize the region first. Then find the function to integrate and the domain over which you need to integrate.

a.

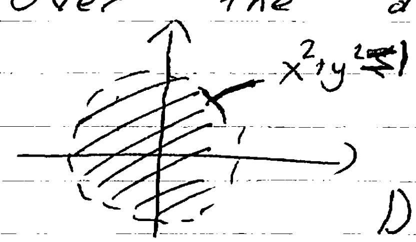


The region is
* outside the cone
* inside the cylinder
* below $z=1$ plane
* above $z=0$ plane.

horizontal ~~vertical~~ cross-sections of the cone and the cylinder on $z=1$ plane is the ~~same~~ circle of radius one.

We need to integrate $z=\sqrt{x^2+y^2}$ as the height of each box in a Riemann sum is given by the equation for the cone.

Over the domain



$D=\{(x,y): x^2+y^2\leq 1\}$

or

in polar coordinates,

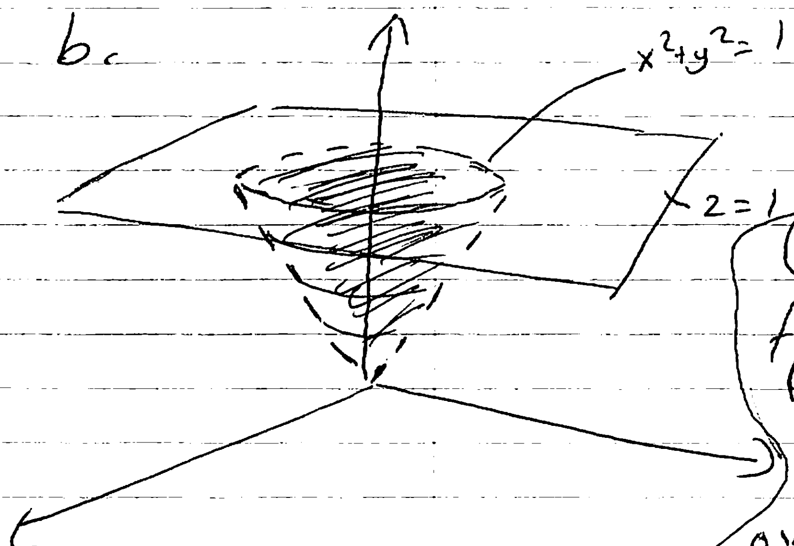
$D=\{(r,\theta): 0\leq r\leq 1, 0\leq\theta\leq 2\pi\}$

$$\text{Volume} = \iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 r\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_{r=0}^{1} d\theta = \int_0^{2\pi} \frac{1}{3}\,d\theta = \frac{2\pi}{3}$$

b.

$x^2+y^2=1$ (Horizontal cross-section of $\sqrt{x^2+y^2}$ on $z=1$)

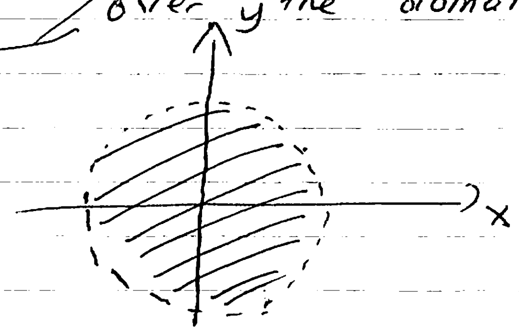


① Function to integrate

$f(x,y) = 1 - \sqrt{x^2+y^2}$

(since the solid is above the cone but below the plane $z=1$)

over the domain



$D = \{(x,y) : x^2+y^2 \le 1\}$

or in polar coordinates

$D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

② Volume $= \iint_D 1 - \sqrt{x^2+y^2}\,dx\,dy$

$$= \int_0^{2\pi}\int_0^1 \left(1 - \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1-r)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3}\right)\Big|_{r=0}^{1} d\theta$$

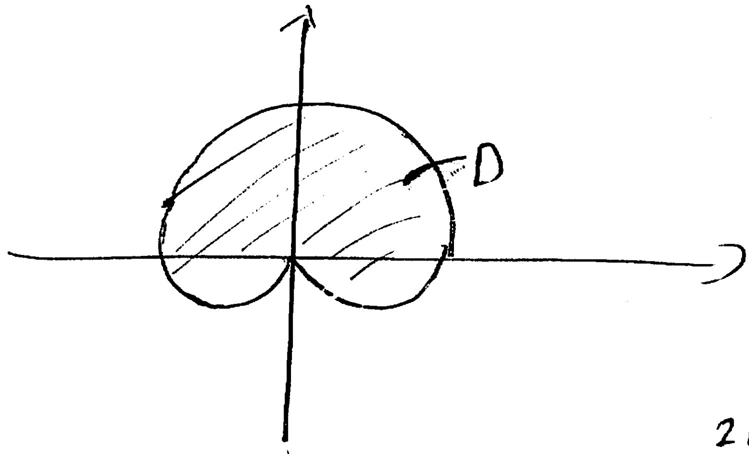
$$= \int_0^{2\pi} \frac{1}{6}\,d\theta = \frac{2\pi}{6} = \frac{\pi}{3}$$

Homework 5
Question 5

Find the area enclosed by the cardioid

$$r(\theta) = 1 + \sin(\theta)$$

by setting up a double integral in polar coordinates.



$D = \{(r,\theta);\ 0 \leq \theta \leq 2\pi,$
$0 \leq r \leq 1 + \sin(\theta)\}$

$$\underbrace{\iint_D 1 \, dx\, dy}_{\substack{\text{Volume of the cylinder with height} \\ \text{one and base area} \\ \text{formed by the cardioid}}} = \int_0^{2\pi}\int_0^{1+\sin\theta} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} \Bigg|_{r=0}^{1+\sin\theta} d\theta$$

=

Area enclosed by the cardioid $= \int_0^{2\pi} \left(\frac{1 + 2\sin(\theta) + \sin^2\theta}{2} \right) d\theta$

$$= \int_0^{2\pi} 1 + \sin\theta - \underbrace{\cos^2\theta}_{\frac{1+\cos(2\theta)}{2}} \, d\theta = \underbrace{\theta - \cos\theta - \frac{\theta}{2} - \frac{\sin(2\theta)}{4}}_{\frac{\theta}{2} - \cos(\theta) - \frac{\sin(2\theta)}{2}} \Bigg|_{\theta=0}^{2\pi}$$

$$\left(\frac{2\pi}{2} - \cos(2\pi)\right) - \left(-\cos(0)\right) = \pi$$