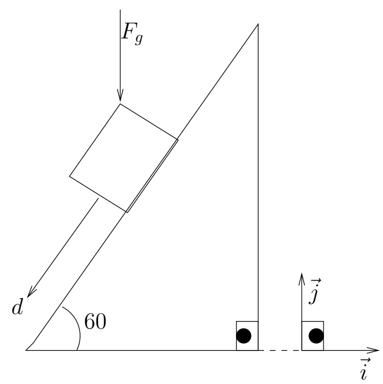
Midterm 1, Math 20C (Lecture C)
November 2nd, 2007

This is the solution set for one of the types of the exam. The solutions to the other type are straightforward modifications of the solutions below.

1. An object slides 5 meters over the inclined surface with slope 60° due to the vertical gravitational force $\mathbf{F_g}$ with magnitude $|\mathbf{F_g}| = 50$ Newton as illustrated in the figure below.



a) (1 point) Express the displacement vector $\mathbf{d}$ (note that $|\mathbf{d}| = 5$ meters) in terms of the standard basis vectors $\vec{i}$ and $\vec{j}$.

Solution:

$$\mathbf{d} = (-|d|\cos(60))\vec{i} + (-|d|\sin(60))\vec{j} = -\frac{5}{2}\vec{i} - \frac{5\sqrt{3}}{2}\vec{j}$$

The signs of both of the components are negative, since the horizontal and vertical components of $\mathbf{d}$ point in the directions opposite to $\vec{i}$ and $\vec{j}$, respectively.

b) (2 points) Find the projection of $\mathbf{F_g}$ onto $\mathbf{d}$.

Solution:
Projection of $\mathbf{F_g}$ onto $\mathbf{d}$ is the vector pointing in the direction of $\mathbf{d}$ with magnitude $|\mathbf{F_g}|\cos(\theta)$ where $\theta = 30$ is the angle between $\mathbf{F_g}$ and $\mathbf{d}$.

$$\text{proj}_{\mathbf{d}}\mathbf{F_g} = |\mathbf{F_g}|\cos(30)\frac{\mathbf{d}}{|\mathbf{d}|} = 25\sqrt{3}\frac{-2.5\vec{i} - 2.5\sqrt{3}\vec{j}}{5} = -12.5\sqrt{3}\vec{i} - 37.5\vec{j}$$

c) (2 points) Find the work done by the gravitational force in Newton $\cdot$ meters.

Solution:
By definition the work done is the dot product of the force and the displacement vectors.

$$\mathbf{W} = \mathbf{F_g} \cdot \mathbf{d} = -50\vec{j} \cdot (-\frac{5}{2}\vec{i} - \frac{5\sqrt{3}}{2}\vec{j}) = -50 \times -\frac{5\sqrt{3}}{2} = 125\sqrt{3}$$

2. The position of a particle in 3D space as a function of time t is given by

$$\mathbf{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad t \geq 0.$$

a) (2 points) Find the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of the particle as functions of time.

Solution:
By definition velocity is the rate of change in the position and acceleration is the rate of change in the velocity.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k} \text{ and } \mathbf{a}(t) = \mathbf{v}'(t) = 2\vec{j} + 6t\vec{k}.$$

b) (2 points) Find the *cosine* of the angle between $\mathbf{a}(1)$ and $\mathbf{v}(1)$.

Solution:
First we evaluate the velocity and acceleration at $t = 1$

$$\mathbf{v}(1) = \vec{i} + 2\vec{j} + 3\vec{k} \text{ and } \mathbf{a}(1) = 2\vec{j} + 6\vec{k}.$$

To find the *cosine* of the angle between $\mathbf{v}(1)$ and $\mathbf{a}(1)$, we equate the algebraic and geometric definitions of the dot product,

$$\mathbf{v}(1) \cdot \mathbf{a}(1) = (1)(0) + (2)(2) + (3)(6) = |\mathbf{v}(1)||\mathbf{a}(1)| \cos(\theta)$$

where θ is the angle between $\mathbf{a(1)}$ and $\mathbf{v(1)}$. Noting that $|\mathbf{v}(1)| = \sqrt{14}$ and $|\mathbf{a}(1)| = \sqrt{40}$, we have

$$\cos(\theta) = \frac{22}{\sqrt{14}\sqrt{40}} = \frac{11}{2\sqrt{35}}.$$

c) (3 points) Find the equation of the plane passing through $\mathbf{r}(1)$ and containing $\mathbf{a}(1)$ and $\mathbf{v}(1)$. This is the *osculating* plane to which the particle stays close around $t = 1$.

Solution:
Since the plane contains the vectors $\mathbf{a}(1)$ and $\mathbf{v}(1)$, any normal vector should be perpendicular to both $\mathbf{a}(1)$ and $\mathbf{v}(1)$. The cross-product of these vectors gives a normal vector for the plane.

$$\begin{aligned}
\mathbf{n} &= \mathbf{v}(1) \times \mathbf{a}(1) = (\vec{\imath} + 2\vec{\jmath} + 3\vec{k}) \times 2\vec{\jmath} + 6\vec{k} \\
&= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \vec{k} = 6\vec{\imath} - 6\vec{\jmath} + 2\vec{k}
\end{aligned}$$

The plane also passes through

$$\mathbf{r}(1) = \vec{\imath} + \vec{\jmath} + \vec{k}.$$

Let $p = (x, y, z)$ be a point on the plane. The displacement vector $\overrightarrow{r(1)p}$ must be perpendicular to the normal vector $\mathbf{n}$. Therefore

$$0 = \mathbf{n} \cdot \overrightarrow{r(1)p} = (6\vec{\imath} - 6\vec{\jmath} + 2\vec{k}) \cdot ((x-1)\vec{\imath} + (y-1)\vec{\jmath} + (z-1)\vec{k}),$$

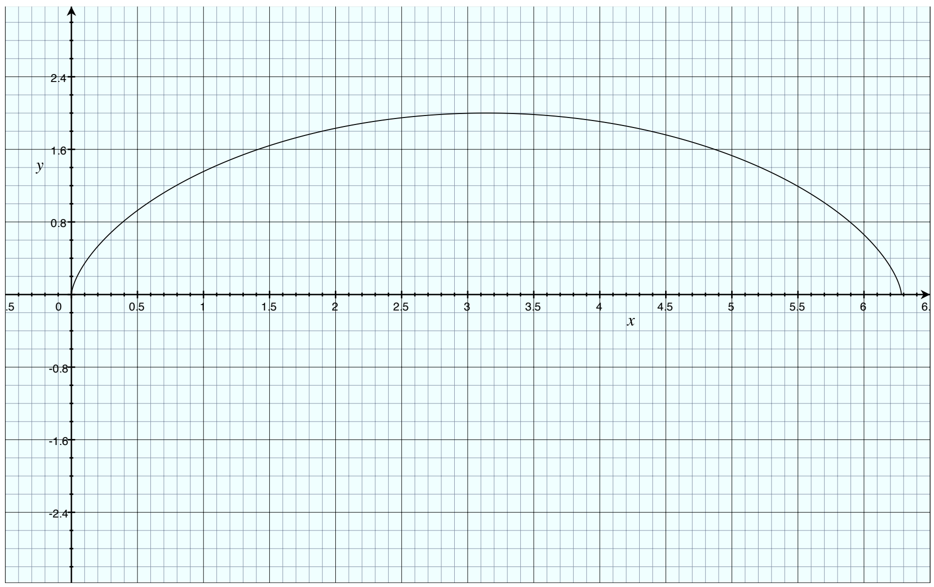
which yields equation for the plane as

$$3x - 3y + z = 1.$$

3. The graph of the cycloid generated by the parametric equation

$$x(\theta) = (\theta - \sin(\theta)) \quad \text{and} \quad y(\theta) = (1 - \cos(\theta)), \qquad 0 \le \theta < 2\pi$$

is illustrated below. The curve intersects the x-axis at $\theta = 0$ and $\theta = 2\pi$.



a) (3 points) Write down a definite integral for the area between the cycloid and the x-axis. Do not evaluate the integral.

Solution:
First we calculate the derivatives.

$$x'(\theta) = 1 - \cos(\theta) \quad \text{and} \quad y'(\theta) = \sin(\theta)$$

The area between the cycloid and the x-axis is given by

$$\int_0^{2\pi} y(\theta)x'(\theta)d\theta = \int_0^{2\pi} (1 - \cos(\theta))(1 - \cos(\theta))d\theta = \int_0^{2\pi} (1 - \cos(\theta))^2 d\theta.$$

The integral above must be positive because x is increasing as a function of θ and y is non-negative for $\theta \in [0, 2\pi)$.

b) (3 points) Find the equation of the line tangent to the cycloid at $\theta = \pi/2$.

Solution:
The slope of the tangent line is given by

$$m = \frac{dy}{dx}\Big|_{\theta=\pi/2} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{\sin(\pi/2)}{1 - \cos(\pi/2)} = 1.$$

The line passes through $(x(\pi/2), y(\pi/2))$ with

$$x(\pi/2) = \pi/2 - \sin(\pi/2) = \pi/2 - 1 \quad \text{and} \quad y(\pi/2) = 1 - \cos(\pi/2) = 1.$$

The equation for the line is

$$\begin{aligned} y - y(1) &= m(x - x(1)) \\ y - 1 &= x - \pi/2 + 1. \end{aligned}$$

4. Consider the parametric curve $\mathcal{C}$ in 3D space

$$x(t) = 3\sin(t), \quad y(t) = 5\cos(t) \quad \text{and} \quad z(t) = 4\sin(t), \ \ t \geq 0$$

a) (3 points) Find the unit tangent vector $\mathbf{T}(t)$ to $\mathcal{C}$.

Solution:
Let $\mathbf{r}(t) = 3\sin(t)\vec{\imath} + 5\cos(t)\vec{\jmath} + 4\sin(t)\vec{k}$ be the vector representation of the curve $\mathcal{C}$. Then

$$\mathbf{r}'(t) = 3\cos(t)\vec{\imath} - 5\sin(t)\vec{\jmath} + 4\cos(t)\vec{k}$$

and

$$|\mathbf{r}'(t)| = \sqrt{(3\cos(t))^2 + (-5\sin(t))^2 + (4\cos(t))^2} = \sqrt{25(\cos^2(t) + \sin^2(t))} = 5.$$

By definition

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{3\cos(t)\vec{\imath} - 5\sin(t)\vec{\jmath} + 4\cos(t)\vec{k}}{5} = 0.6\cos(t)\vec{\imath} - \sin(t)\vec{\jmath} + 0.8\cos(t)\vec{k}.$$

b) (2 points) Find the unit tangent vector $\hat{\mathbf{T}}(s)$ parametrized in terms of the arc-length s, that is $\hat{\mathbf{T}}(s) = \mathbf{T}(t')$ where s is the arc-length of the curve $\mathcal{C}$ from $t = 0$ to $t = t'$.

Solution:
First we need to find the arc-length s of $\mathcal{C}$ from 0 to t',

$$s = f(t') = \int_0^{t'} |r'(t)|dt = \int_0^{t'} 5dt = 5t'.$$

The arc-length parametrization $\hat{\mathbf{T}}(s)$ and $\mathbf{T}(t')$ are related by

$$\hat{\mathbf{T}}(s) = \mathbf{T}(f^{-1}(s)) = \mathbf{T}(s/5) = \mathbf{T}(t').$$

Above $f^{-1}(s) = s/5 = t'$ is the inverse of $f(t') = 5t'$, that is the function $f^{-1}(s)$ returns the t value where the arc-length is s. Therefore the arc-length parametrization of the tangent vector is given by

$$\hat{\mathbf{T}}(s) = \mathbf{T}(s/5) = 0.6\cos(s/5)\vec{\imath} - \sin(s/5)\vec{\jmath} + 0.8\cos(s/5)\vec{k}.$$

c) (2 points) Show that the curvature $\kappa(s)$ of $\mathcal{C}$ is constant and equal to $1/5$.

Solution:
Differentiating $\hat{\mathbf{T}}(s)$ component-wise yields

$$\hat{\mathbf{T}}'(s) = \frac{-0.6\sin(s/5)\vec{i} - \cos(s/5)\vec{j} - 0.8\sin(s/5)\vec{k}}{5}.$$

By definition

$$\kappa(s) = |\hat{\mathbf{T}}'(s)| = \frac{\sqrt{(-0.6\sin(s/5))^2 + (-\cos(s/5))^2 + (-0.8\sin(s/5))^2}}{5} = 1/5.$$