Midterm 2, Math 20C (Lecture C)
November 28th, 2007

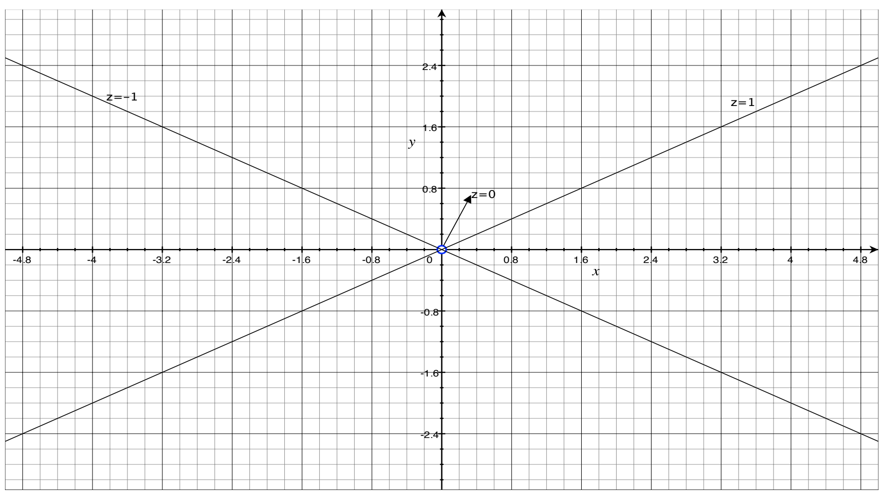
1. Consider the function

$$f(x,y) = \begin{cases} \frac{x^2+4y^2}{4xy} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

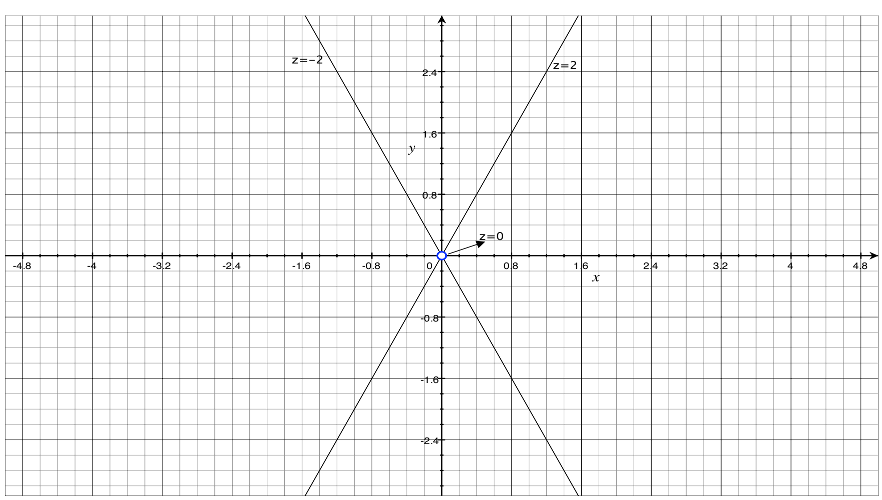
a) (3 points) Plot the contour diagrams of $z = f(x,y)$ for $z = -1, 0, 1$.

Setting $z = -1$, we have $0 = x^2 + 4xy + 4y^2 = (x+2y)^2$.
Setting $z = 1$, we have $0 = x^2 - 4xy + 4y^2 = (x-2y)^2$.
The function takes the value $z = 0$, only at $(x,y) = (0,0)$.

Therefore the level sets when $z = -1$ and $z = 1$ are the lines $x = -2y$ and $x = 2y$, respectively. The level set $z = 0$ is just the origin.



Answer for the other version: The level sets when $z = -2$ and $z = 2$ are the lines $y = -2x$ and $y = 2x$, respectively. The level set $z = 0$ is just the origin.



b) (3 points) Is $f(x,y)$ continuous at $(x,y)=(0,0)$? Justify your answer.

No, it is not continuous at $(0,0)$, because

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist. Suppose we approach the origin along the line $y=kx$. The function along this line is $f(x,kx)=\frac{x^2+4k^2x^2}{4kx^2}=\frac{1+4k^2}{4k}$. For instance along the line $y=x$ $(k=1)$, the function is constant and equal to $\frac{5}{4}$. But along the line $y=-x$ $(k=-1)$, the function takes the value $-\frac{5}{4}$. Since along different paths leading to $(0,0)$ the function $f(x,y)$ approaches different values, the limit as $(x,y)\to(0,0)$ does not exist.

Answer for the other version: The function is discontinuous at $(0,0)$, since the limit as $(x,y)\to(0,0)$ does not exist. Along different lines, the function approaches different values as $(x,y)\to(0,0)$.

2. The kinetic energy of an object in *Joules* (a unit for work and energy) with mass m in kg and velocity v in m/sec is given by the equation

$$E(m,v)=\frac{1}{2}mv^2.$$

a) (3 points) Find the linearization of $E(m,v)$ at $(m,v)=(3,6)$.

We need to evaluate the function and its derivatives at $(3,6)$.

$$\begin{aligned} E(3,6) &= 0.5(3)(6)^2=54 \\ E_1(x,y) &= \frac{1}{2}v^2 \Rightarrow E_1(3,6)=0.5(6^2)=18 \\ E_2(x,y) &= mv \Rightarrow E_2(3,6)=(3)(6)=18 \end{aligned}$$

The linearization of $E(m,v)$ at $(3,6)$ is the equation of the tangent plane to $E(m,v)$ at $(3,6)$

$$L(m,v)=E(3,6)+E_1(3,6)(m-3)+E_2(3,6)(v-6)$$
$$L(m,v)=54+18(m-3)+18(v-6).$$

Answer for the other version: The linearization of $E(m,v)$ at $(1,2)$ is

$$L(m,v)=E(1,2)+E_1(1,2)(m-1)+E_2(1,2)(v-2)$$
$$L(m,v)=2+2(m-1)+2(v-2).$$

b) (3 points) The mass and velocity of an object are measured as 3 kg and 6 m/sec, respectively. If each of the actual values of the mass and velocity is %1 greater than the corresponding measured value, estimate the kinetic energy of the object using the linearization from part **a)**.

The actual mass and velocity of the object are $m = 1.01(3) = 3.03$ and $v = 1.01(6) = 6.06$. Substituting these values in the linearization will give an approximate value for the kinetic energy.

$$L(3.03, 6.06) = 54 + 18(0.03) + 18(0.06) = 55.62\ Joules.$$

Answer for the other version: The actual values of mass and velocity are $m = 1.01$ and $v = 2.02$.

$$L(1.01, 2.02) = 2 + 2(0.01) + 2(0.02) = 2.06\ Joules.$$

3. Let

$$g(x, y) = e^{(x-1)^2 - y^2}$$

a) (2 points) Find the gradient vector $\nabla g(2, 1)$.

The partial derivatives of g with respect to x and y are given by

$$g_1(x, y) = 2(x-1)e^{(x-1)^2-y^2} \text{ and } g_2(x, y) = -(2y)e^{(x-1)^2-y^2}$$

We need to evaluate the partial derivatives at $(2, 1)$.

$$\nabla g(2, 1) = (g_1(2, 1), g_2(2, 1)) = (2, -2).$$

Answer for the other version: The partial derivatives are given by

$$g_1(x, y) = (2x)e^{x^2-(y-2)^2} \text{ and } g_2(x, y) = -2(y-2)e^{x^2-(y-2)^2}.$$

Evaluating the derivatives at $(1, 3)$ yields

$$\nabla g(1, 3) = (2, -2).$$

b) (3 points) Find the directional derivative $D_{\vec{u}}g(2,1)$ in the direction of $\vec{u} = \frac{1}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j}$.

The directional derivative is the dot product of the gradient and the direction vectors.

$$D_{\vec{u}}g(2,1) = \nabla g(2,1) \cdot \vec{u} = (2,-2) \cdot (\frac{1}{2}, -\frac{\sqrt{3}}{2}) = 2(1/2) + (-2)(-\sqrt{3}/2) = 1 + \sqrt{3}.$$

Answer for the other version:

$$D_{\vec{u}}g(1,3) = \nabla g(1,3) \cdot \vec{u} = (2,-2) \cdot (-3/5, -4/5) = 2(-3/5) + (-2)(-4/5) = 2/5.$$

c) (3 points) Find the unit vector $\vec{u}$ so that the directional derivative $D_{\vec{u}}g(2,1)$ in the direction of $\vec{u}$ at $(x,y) = (2,1)$ is as small as possible, that is for any unit vector $\vec{w}$, $D_{\vec{u}}g(2,1) \leq D_{\vec{w}}g(2,1)$.

Once again using the dot product formula for the directional derivatives

$$D_{\vec{u}}g(2,1) = \nabla g(2,1) \cdot \vec{u} = |\nabla g(2,1)| \cos(\theta)$$

where θ is the angle between the direction vector $\vec{u}$ and $\nabla g(2,1)$. If we choose $\theta = \pi$, $\cos(\theta) = -1$ and $D_{\vec{u}}g(2,1) = -|\nabla g(2,1)|$ is as small as possible. Therefore the directional derivative is minimized when $\vec{u}$ points in the direction opposite to $\nabla g(2,1)$, that is

$$\vec{u} = -\frac{\nabla g(2,1)}{|\nabla g(2,1)|} = -\frac{(2,-2)}{2\sqrt{2}} = (-1/\sqrt{2}, 1/\sqrt{2}).$$

Answer for the other version: The directional derivative is minimized in the direction opposite to $\nabla g(1,3)$.

$$\vec{u} = -\frac{\nabla g(1,3)}{|\nabla g(1,3)|} = -\frac{(2,-2)}{2\sqrt{2}} = (-1/\sqrt{2}, 1/\sqrt{2}).$$

4. (5 points) Find the points on the curve $yx = 16$ that are closest to the origin by posing a constrained optimization problem and solving it using Lagrange multipliers. (Remark : The global minima of $x^2 + y^2$ and $\sqrt{x^2 + y^2}$ subject to any constraint are the same.)

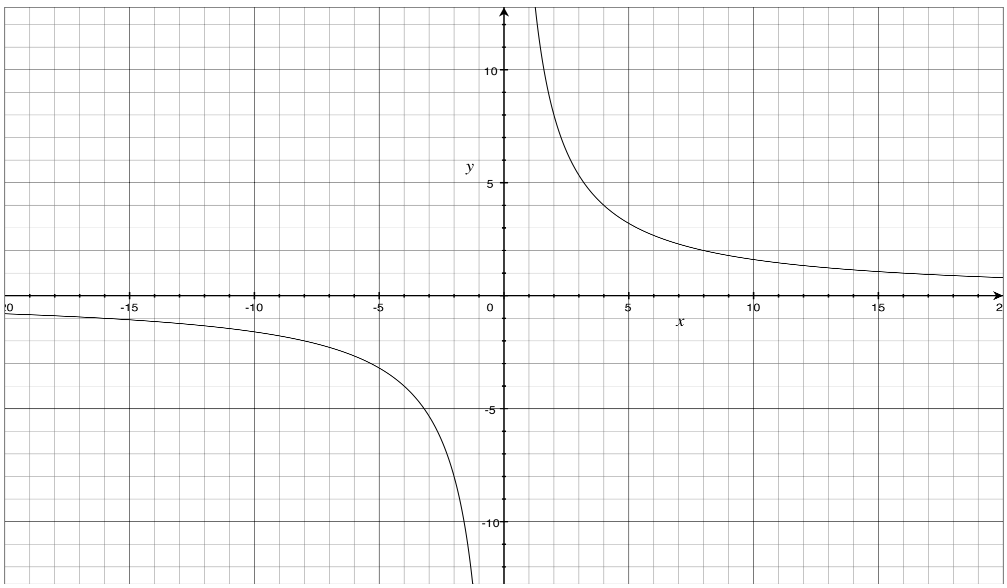


Figure 1: The graph of the curve $yx = 16$ is illustrated above.

The constrained optimization problem can be posed as

$$\text{minimize} \;\; f(x, y) \;\; \text{subject to} \;\; g(x, y) = 16$$

where the objective and constraints functions are

$$f(x, y) = \sqrt{x^2 + y^2} \;\; \text{and} \;\; g(x, y) = yx,$$

respectively. Furthermore we can minimize $h(x, y) = f^2(x, y) = x^2 + y^2$ instead of $f(x, y) = \sqrt{x^2 + y^2}$, because both of the functions must be minimized at exactly the same point. (This is to simplify the derivatives; you would obtain the same answer, if you minimize $f(x, y)$, but the solution would be longer.) Therefore we need to solve the problem

$$\text{minimize} \;\; h(x, y) \;\; \text{subject to} \;\; g(x, y) = 16$$

We need to calculate the gradients of $h(x, y)$ and $g(x, y)$.

$$\nabla h(x, y) = (2x, 2y) \;\; \text{and} \;\; \nabla g(x, y) = (y, x)$$

At the closest point the gradient vectors must be multiples of each other,

$$\nabla h(x, y) = \lambda \nabla g(x, y) \Rightarrow (i)\; 2x = \lambda y, \;\; (ii)\; 2y = \lambda x.$$

By eliminating λ in the rightmost two equations, we have

$$\frac{2x}{y} = \frac{2y}{x} \Rightarrow x = \pm y.$$

But $x = -y$ does not satisfy the constraint $xy = 16$; if we substitute $-y$ for x in the constraint, we obtain $-y^2 = 16$. By substituting y for x in the constraint we obtain

$$y^2 = 16 \Rightarrow y = \pm 4.$$

Therefore the closest points on the curve $yx = 16$ are

$$(4, 4) \text{ and } (-4, -4).$$

Answer for the other version: The solution above with the constraint $yx = 4$ yields the closest points on the curve as

$$(2, 2) \text{ and } (-2, -2).$$