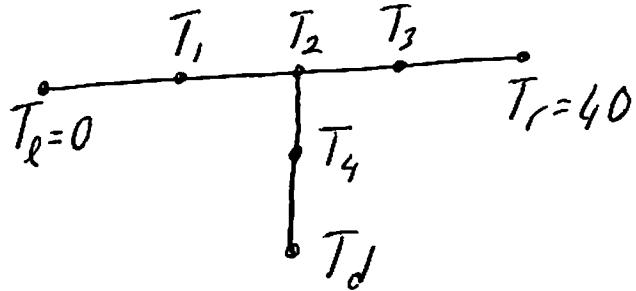


Solutions to Homework 1

1.



(a) $T_d = 50$ is given.

Equation for the first node $T_1 = \frac{T_2 + 0}{2} \Rightarrow 2T_1 - T_2 = 0$

Equation for the second node $T_2 = \frac{T_1 + T_3 + T_4}{3} \Rightarrow T_1 - 3T_2 + T_3 + T_4 = 0$

Equation for the third node $T_3 = \frac{T_2 + 40}{2} \Rightarrow T_2 - 2T_3 = -40$

Equation for the fourth node $T_4 = \frac{T_2 + T_d}{2} \Rightarrow T_2 - 2T_4 = -50$

Solve the linear system

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -40 \\ -50 \end{bmatrix}$$

Reduce the Augmented matrix into the reduced echelon form

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & -50 \end{array} \right] \xrightarrow{\text{(1)} T_1 \leftrightarrow T_2} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & -50 \end{array} \right] \xrightarrow{\text{(2)} T_2 := T_2 - 2T_1} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & -50 \end{array} \right]$$

$$\begin{array}{l}
 (3) T_2 \leftrightarrow T_3 \\
 (4) T_3 := T_3 - 5T_2 \\
 \xrightarrow{(5) T_4 := T_4 - T_2} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 8 & -2 & 200 \\ 0 & 0 & 2 & -2 & -10 \end{array} \right] \\
 \xrightarrow{(6) T_3 := T_3/2} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 4 & -1 & 100 \\ 0 & 0 & 1 & -1 & -5 \end{array} \right] \\
 \xrightarrow{(7) T_4 := T_4/2} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 2 & -1 & 50 \\ 0 & 0 & 1 & -1 & -5 \end{array} \right]
 \end{array}$$

$$\begin{array}{l}
 (8) T_3 \leftrightarrow T_4 \\
 (9) T_4 := T_4 - 4T_3 \\
 \xrightarrow{\text{echelon form}} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 3 & 120 \end{array} \right] \\
 \xrightarrow{(10) T_4 := T_4/3} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right] \\
 \xrightarrow{(11) T_3 := T_3 + T_4} \left[\begin{array}{ccccc} 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 0 & 1 & 1 & 35 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right] \\
 \xrightarrow{(12) T_1 := T_1 - T_4} \left[\begin{array}{ccccc} 1 & -3 & 1 & 0 & 15 \\ 0 & 1 & -2 & 0 & 30 \\ 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right]
 \end{array}$$

$$\begin{array}{l}
 (13) T_2 := T_2 + 2T_3 \\
 (14) T_1 := T_1 - T_3 \\
 \xrightarrow{\text{echelon form}} \left[\begin{array}{ccccc} 1 & -3 & 0 & 0 & -75 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right] \\
 \xrightarrow{(15) T_1 := T_1 + 3T_2} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 30 \\ 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 1 & 40 \end{array} \right] \\
 \boxed{T_1 = 15, \quad T_2 = 30, \quad T_3 = 35, \quad T_4 = 40}
 \end{array}$$

(b) T_d is unknown.

We have a linear system of 4 equations in 5 unknowns.

Linear System

$$2T_1 - T_2 = 0$$

$$T_1 - 3T_2 + T_3 + T_4 = 0$$

$$T_2 - 2T_3 = -40$$

$$T_2 - 2T_4 + T_d = 0$$

$$\begin{array}{c}
 \text{Matrix equation} \\
 \left[\begin{array}{ccccc} 2 & -1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 \end{array} \right] \left[\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_d \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]
 \end{array}$$

Reduce the augmented matrix into the reduced echelon form

$$\begin{array}{l}
 \left[\begin{array}{ccccc} 2 & -1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & 0 \end{array} \right] \\
 \xrightarrow{(1) T_1 \leftrightarrow T_2} \left[\begin{array}{ccccc} 1 & -3 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & 0 \end{array} \right] \\
 \xrightarrow{(2) T_2 := T_2 - 2T_1} \left[\begin{array}{ccccc} 1 & -3 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 0 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & 0 \end{array} \right]
 \end{array}$$

2

$$\begin{array}{l}
 (3) T_2 \leftrightarrow T_3 \\
 (4) T_3 := T_3 - 5T_2 \\
 \hline
 (5) T_4 := T_4 - T_2
 \end{array}
 \left[\begin{array}{cccccc}
 1 & -3 & 1 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & 0 & -40 \\
 0 & 0 & 8 & -2 & 0 & 200 \\
 0 & 0 & 2 & -2 & 1 & 40
 \end{array} \right]
 \xrightarrow{\text{(6) } T_3 := T_3/2}
 \left[\begin{array}{cccccc}
 1 & -3 & 1 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & 0 & -40 \\
 0 & 0 & 4 & -1 & 0 & 100 \\
 0 & 0 & 2 & -2 & 1 & 40
 \end{array} \right]$$

$$\begin{array}{l}
 (7) T_3 \leftrightarrow T_4 \\
 (8) T_4 := T_4 + 2T_3 \\
 \hline
 (9) T_4 := T_4/3
 \end{array}
 \left[\begin{array}{cccccc}
 1 & -3 & 1 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & 0 & -40 \\
 0 & 0 & 2 & -2 & 1 & 40 \\
 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{20}{3}
 \end{array} \right]
 \xrightarrow{\text{(10) } T_3 := T_3 + 2T_4}
 \left[\begin{array}{cccccc}
 1 & -3 & 1 & 0 & \frac{2}{3} & -\frac{20}{3} \\
 0 & 1 & -2 & 0 & 0 & -40 \\
 0 & 0 & 2 & 0 & \frac{1}{3} & \frac{160}{3} \\
 0 & 0 & 0 & 1 & \frac{-2}{3} & \frac{20}{3}
 \end{array} \right]$$

$$\begin{array}{l}
 (12) T_3 := T_3/2 \\
 (13) T_2 := T_2 + 2T_3 \\
 \hline
 (14) T_1 := T_1 - T_3
 \end{array}
 \left[\begin{array}{cccccc}
 1 & -3 & 0 & 0 & \frac{5}{6} & \frac{100}{3} \\
 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{40}{3} \\
 0 & 0 & 1 & 0 & -\frac{1}{6} & \frac{80}{3} \\
 0 & 0 & 0 & 1 & \frac{-2}{3} & \frac{20}{3}
 \end{array} \right]
 \xrightarrow{\text{(15) } T_1 := T_1 + T_2}
 \left[\begin{array}{cccccc}
 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{20}{3} \\
 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{40}{3} \\
 0 & 0 & 1 & 0 & -\frac{1}{6} & \frac{80}{3} \\
 0 & 0 & 0 & 1 & -\frac{2}{3} & \frac{20}{3}
 \end{array} \right]$$

From the last row $T_4 = \frac{20}{3} + \frac{2}{3} T_d$

From the third row $T_3 = \frac{80}{3} + \frac{1}{6} T_d$

From the second row $T_2 = \frac{40}{3} + \frac{1}{3} T_d$

From the first row $T_1 = \frac{20}{3} + \frac{1}{6} T_d$

Solution set

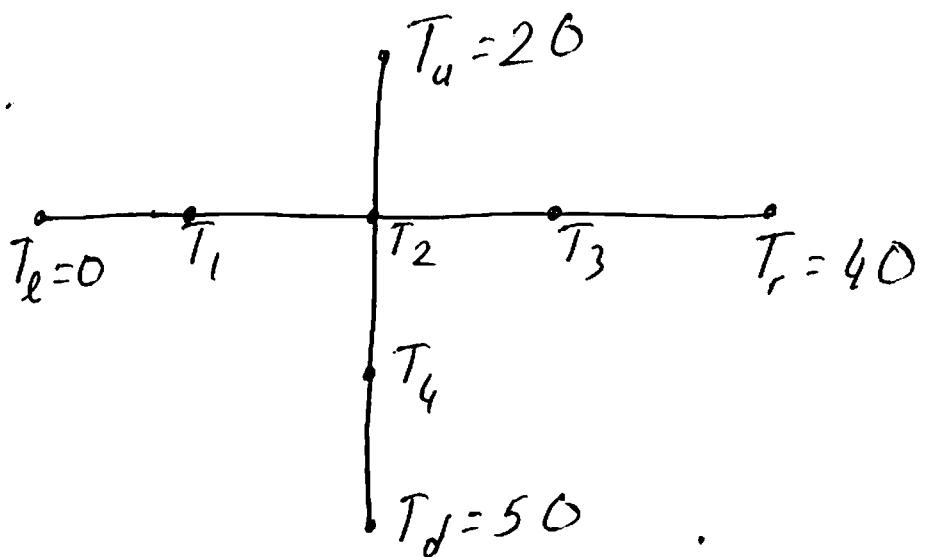
$$\left\{ \begin{array}{l}
 T_1 = \frac{20}{3} + \frac{T_d}{6} \\
 T_2 = \frac{40}{3} + \frac{T_d}{3} \\
 T_3 = \frac{80}{3} + \frac{T_d}{6} \\
 T_4 = \frac{20}{3} + \frac{2T_d}{3} \\
 T_d \text{ is free}
 \end{array} \right.$$

In set notation

$$\text{Solution set} = \left\{ \left[\begin{array}{c} \frac{20}{3} \\ \frac{40}{3} \\ \frac{80}{3} \\ \frac{20}{3} \end{array} \right] + T_d \left[\begin{array}{c} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{2}{3} \end{array} \right] : T_d \in \mathbb{R} \right\}$$

(3)

2.



Equation for the first node $T_1 = \frac{T_2 + 0}{2} \Rightarrow 2T_1 - T_2 = 0$

Equations for the second node $T_2 = \frac{T_1 + T_3}{2} \Rightarrow T_1 - 2T_2 + T_3 = 0$

$$T_2 = \frac{20 + T_4}{2} \Rightarrow 2T_2 - T_4 = 20$$

Equations for the third node $T_3 = \frac{40 + T_2}{2} \Rightarrow T_2 - 2T_3 = -40$

Equation for the fourth node $T_4 = \frac{50 + T_2}{2} \Rightarrow T_2 - 2T_4 = -50$

Verify that the linear system

over-determined system

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ -40 \\ -50 \end{bmatrix}$$

is inconsistent. Reduce the augmented matrix into echelon form.

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 20 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & -50 \end{array} \right] \xrightarrow{\begin{array}{l} (1) T_1 \leftrightarrow T_2 \\ (2) T_2 := T_2 - 2T_1 \end{array}} \left[\begin{array}{ccccc} 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 20 \\ 0 & 1 & -2 & 0 & -40 \\ 0 & 1 & 0 & -2 & -50 \end{array} \right]$$

(4)

$$\begin{array}{l}
 (3) T_2 \leftrightarrow T_4 \\
 (4) T_3 := T_3 - 2T_2 \\
 \xrightarrow{(5) T_4 := T_4 - 3T_2} \\
 (6) T_5 := T_5 - T_2
 \end{array}
 \left[\begin{array}{ccccc}
 1 & -2 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & -40 \\
 0 & 0 & 4 & -1 & 100 \\
 0 & 0 & 4 & 0 & 120 \\
 0 & 0 & 2 & -2 & -10
 \end{array} \right]
 \begin{array}{l}
 (7) r_4 := r_4 / 4 \\
 (8) r_5 := r_5 / 2
 \end{array}
 \left[\begin{array}{ccccc}
 1 & -2 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & -40 \\
 0 & 0 & 1 & -\frac{1}{4} & 25 \\
 0 & 0 & 1 & 0 & 60 \\
 0 & 0 & 1 & -1 & -5
 \end{array} \right]$$

$$\begin{array}{l}
 (9) \underline{r_3 \leftrightarrow r_4} \\
 (10) \underline{r_4 := r_4 - 4r_3} \\
 (11) \underline{r_5 := r_5 - r_3}
 \end{array}
 \left[\begin{array}{ccccc}
 1 & -2 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & -40 \\
 0 & 0 & 1 & 0 & 30 \\
 0 & 0 & 0 & -1 & -20 \\
 0 & 0 & 0 & -1 & -35
 \end{array} \right]
 \begin{array}{l}
 (12) r_5 := r_5 - r_4
 \end{array}
 \left[\begin{array}{ccccc}
 1 & -2 & 1 & 0 & 0 \\
 0 & 1 & -2 & 0 & -40 \\
 0 & 0 & 1 & 0 & 30 \\
 0 & 0 & 0 & -1 & -20 \\
 0 & 0 & 0 & 0 & -15
 \end{array} \right]$$

The system is inconsistent,
since the last column in the
echelon form is a pivot column.

last
column
is a
pivot
column

echelon form

3.

(a) This is obvious.

Consider the systems

$$\begin{array}{l}
 a_{11}x_1 + a_{12}x_2 = b_1, \quad \text{and} \quad a_{11}x_1 + a_{12}x_2 = b_1, \\
 a_{21}x_1 + a_{22}x_2 = b_2 \quad \xrightarrow{\text{System obtained by row-replacement}}
 \end{array}
 \begin{array}{l}
 (a_{21} + a_{11})x_1 + (a_{22} + a_{12})x_2 = b_2 + cb,
 \end{array}$$

Original system

* (x_1, x_2) satisfies the original system

$$a_{11}x_1 + a_{12}\cancel{x_2} \xrightarrow{\text{holds}} b_1, \quad \text{holds}$$

First equation of the altered system
is satisfied by (x_1, x_2) .

- * Since (x_1, x_2) satisfies both of the equations of the original system, it also satisfies both of the equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$(a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 = cb_2$$

By adding equations side by side

$$(a_{11} + ca_{21})x_1 + (a_{12} + ca_{22})x_2 = b_1 + cb_2$$

Second equation of the altered system
is also satisfied by (x_1, x_2) .

Therefore (x_1, x_2) is a solution for the altered system.

- (b) The reverse also hold, since the row-replacement operation is reversible.

Formally assume (x_1, x_2) is a solution for the ~~altered~~ system

altered system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ (a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 = b_2 + cb_1 \end{cases}$

- * The first equation of the original system is identical to the original system above. The point (x_1, x_2) satisfies the first equation of the original system.

- * Multiply the first equation of the altered system by c and subtract it from the second equation to obtain

$$a_{21}x_1 + a_{22}x_2 = b_2$$

The point (x_1, x_2) satisfies the second equation of the original system

Therefore the point (x_1, x_2) is a solution for the original system.

Part (a) and (b) combined show that

$\boxed{\begin{array}{l} (x_1, x_2) \text{ is a solution of the original system} \\ \xleftarrow{\hspace{2cm}} (x_1, x_2) \text{ is a solution of the system obtained by row-replacement.} \end{array}}$

4. For all parts we need to reduce the matrix A into an echelon form.

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} (1) r_3 := r_3 - r_1 \\ (2) r_4 := r_4 - r_1 \end{array}} \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{array} \right]$$

$$\begin{array}{l} (3) r_2 \leftrightarrow r_4 \\ (4) r_3 := r_3 - \frac{3}{2}r_2 \\ \hline (5) r_4 := r_4 + \frac{3}{2}r_2 \end{array} \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array} \right] \xrightarrow{(6) r_4 := r_4 + r_3} \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

pivot positions
echelon matrix

(a) All of the columns are pivot columns.
The positions $(1,1)$, $(2,2)$ and $(3,3)$ are pivot positions.

(b) Whether columns of A span \mathbb{R}^4 , is equivalent to

is the system

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

consistent for all $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{R}^4$?

But the echelon form for the augmented matrix will look like

$$\begin{array}{cccc|c} 1 & 2 & 0 & b_1 \\ 0 & 3 & -1 & b_2 \\ 1 & -1 & 1 & b_3 \\ 1 & 0 & 0 & b_4 \end{array} \xrightarrow{\text{row reduction}} \begin{array}{cccc|c} 1 & 2 & 0 & c_1 \\ 0 & -2 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \\ 0 & 0 & 0 & c_4 \end{array}$$

echelon
form for
 A

By choosing b_1, b_2, b_3, b_4 one can make $c_4 \neq 0$. Therefore the ~~system~~ last column can be a pivot column meaning the system can be inconsistent.

Therefore the columns of A
do not span \mathbb{R}^4

(c) ~~No~~, the columns of A are linearly independent.

Since Augmented matrix for $Ax=0$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & +3 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Echelon form for A

implying there are no free variables.

\Rightarrow The system $Ax=0$ has a unique solution, which must be $x=0$. (the trivial solution)

The vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

is satisfied only for $x_1 = x_2 = x_3 = 0$.
(No non-zero solution satisfies the vector equation)

$$\left[\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right] \text{ are linearly independent.}$$

5.

(a) The solution for the system $Ax=b$ is in the form

$$\boxed{v = v_p + v_h}$$

- * v_p is a particular solution satisfying $Av_p=b$
- * v_h is a solution for the homogeneous system satisfying $Ax=0$.

$$\boxed{v_p = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ is given, } v_h \in \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

For instance

$$\boxed{v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}}$$

is a solution

(b) ~~The~~ The vector v is a solution for $Ax=b$ if and only if

$$\boxed{\frac{v - v_p}{v_h} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

$$v - v_p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The system

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is inconsistent, since

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 := R_2 - R_1 \\ R_3 := R_3 - R_1 \end{array}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 := R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

lost column
is a pivot column

10

Therefore the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has no solution.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

7. (a) Span of two vectors in \mathbb{R}^2

Clearly for $v_1 = 0, v_2 = 0$

$$\begin{aligned} \text{span}\{v_1, v_2\} &= \left\{ c_1 \underbrace{v_1}_0 + c_2 \underbrace{v_2}_0 : c_1, c_2 \in \mathbb{R} \right\} \\ &= \{0\} \end{aligned}$$

For no other pair of vectors $v_1, v_2 \in \mathbb{R}^2$

$$\text{span}\{v_1, v_2\} = 0$$

holds. Without loss of generality assume $v_1 \neq 0$. Now choose $c_1 \neq 0$ and $c_2 = 0$

$$c_1 v_1 + c_2 v_2 = c_1 v_1 \in \text{span}\{v_1, v_2\}.$$

That is if either $v_1 \neq 0$ or $v_2 \neq 0$, span of v_1 and v_2 contains non-zero vectors.

(b) A pair of vectors v_1, v_2 span a line if and only if one of them is a multiple of the other, that is

$$v_2 = \alpha v_1 \quad \text{or} \quad v_1 = \alpha v_2 \quad \begin{array}{l} \text{(and not} \\ \text{both } v_1 \text{ and } v_2 \\ \text{are 0)} \end{array}$$

Formally this condition is equivalent to

$$\alpha v_1 - v_2 = 0 \quad \text{or} \quad v_1 - \alpha v_2 = 0.$$

Therefore v_1, v_2 are linearly dependent

For instance the span of

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

is

$$\begin{aligned} & \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -4 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ (c_1 - 2c_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \end{aligned}$$

which is a line.

(c) $\boxed{\text{A pair of vectors } v_1, v_2 \in \mathbb{R}^2 \text{ span } \mathbb{R}^2 \text{ if and only if they are not multiples of each other.}}$

Formally this means for no α either of $v_1 + \alpha v_2 = 0$ and $v_2 + \alpha v_1 = 0$ is satisfied. That is v_1 and v_2 are linearly independent.

The vectors

span \mathbb{R}^2 . For any $b \in \mathbb{R}^2$ consider the system

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix},$$

which is consistent for all b_1 and b_2 . Therefore

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

has a solution for all $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. That is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ span the whole space.

(12)

(d) There are no other possibilities for the span of two vectors. Span generates a set that is closed under addition and multiplication with scalars.

Closedness under addition: $v, w \in \text{span}\{v_1, v_2\}$
 $\implies v+w \in \text{span}\{v_1, v_2\}$

Closedness under multiplication: $v \in \text{span}\{v_1, v_2\} \implies cv \in \text{span}\{v_1, v_2\}$
with scalars

The only subsets of \mathbb{R}^2 closed under addition and multiplication with scalars is

- * $\{0\}$
- * A line passing through the origin
- * \mathbb{R}^2 (whole space).

6.

(a) Using the constraints that the polynomial has to pass through $(1, 2)$ and $(-1, 4)$, we have the linear equations

$$\begin{aligned} p_2(1) &= a_1 + a_2(1) + a_3(1)^2 = 2 \\ p_2(-1) &= a_1 + a_2(-1) + a_3(-1)^2 = 4 \end{aligned}$$

with a_1 , a_2 and a_3 unknowns. The augmented matrix for this system is

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 4 \end{array} \right].$$

(b) Subtracting the first row of the augmented matrix from the second, and dividing the second row by -2 yields the echelon matrix

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array} \right].$$

Subtracting the second row of the matrix above from the first yields the reduced echelon matrix

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array} \right].$$

From the reduced echelon matrix we deduce that the system is consistent (as the last column is not a pivot column), but the solution is not unique (as a_3 is a free variable). From the reduced echelon matrix, any polynomial that passes through $(1, 2)$ and $(-1, 4)$ must satisfy

$$\begin{aligned} a_2 &= -1 \\ a_1 + a_3 &= 3. \end{aligned}$$

Therefore any triple in the form

$$\left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right] = \left[\begin{array}{c} 3-t \\ -1 \\ t \end{array} \right]$$

where t is any real number is a solution. In particular by setting $t = 1$ ($a_1 = 2$, $a_2 = -1$, $a_3 = 1$) we see that the polynomial $2 - x + x^2$ passes through the given points.

(c) Plugging $x = 1$ and $x = -1$ in the equation for $q_2(x)$ yields

$$v_{a_2, a_3} = \left[\begin{array}{c} a_2 + a_3 \\ -a_2 + a_3 \end{array} \right] = a_2 \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + a_3 \left[\begin{array}{c} 1 \\ 1 \end{array} \right].$$

Since a_2 and a_3 can be any pair of real numbers, all such v_{a_2, a_3} consist of

$$\text{span} \left\{ \left[\begin{array}{c} 1 \\ -1 \end{array} \right], \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\}.$$

Furthermore, as you explore in question 7. the span of two vectors each of size 2 is \mathbb{R}^2 unless one of the vectors is a multiple of the other, which is the case above.