

Solutions to Hw 2

2.1.2

$$\begin{aligned}
 A + 2B &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + 2 \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} \\
 &= \begin{bmatrix} 16 & -10 & +1 \\ 6 & -13 & -4 \end{bmatrix} \quad \begin{array}{l} \text{addition} \\ \text{entry-wise} \end{array}
 \end{aligned}$$

multiply
entry-wise
by 2

$$3C - E = 3 \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} -5 \\ 3 \end{bmatrix} \quad \text{is undefined}$$

You can only add two matrices with same dimension.

$$\begin{aligned}
 CB &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\
 &= \left[\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \right] \\
 &= \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}
 \end{aligned}$$

$$EB = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \underbrace{\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}}_{2 \times 3} \quad \text{is not defined.}$$

since col # of E
 \neq
row # of B

2.1.6

Column-wise

$$AB = [Ab_1 \ Ab_2]$$

$$= \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

Entry-wise

$$AB = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} [b_1 \ b_2] = \begin{bmatrix} \bar{a}_1 b_1 & \bar{a}_1 b_2 \\ \bar{a}_2 b_1 & \bar{a}_2 b_2 \\ \bar{a}_3 b_1 & \bar{a}_3 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} [4 \ 2][1] & [4 \ 2][3] \\ [3 \ 0][1] & [3 \ 0][3] \\ [3 \ 5][1] & [3 \ 5][3] \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

2.1.25.

To see that $n=m$, Suppose

$$Ax = 0 \quad \text{for some } x \in \mathbb{R}^n$$

Since

$$CA = I_n \implies \underbrace{CAx}_0 = I_n x = x$$

$$\implies x = 0$$

Therefore $Ax=0$ is satisfied only for $x=0$.

Columns of A are lin. ind.

(2)

This implies $m \geq n$, as otherwise
 $(\text{when } m < n)$ n vectors in \mathbb{R}^m are lin. dep.

Similarly $AD = I_m$ implies that the columns
of $D_{n \times m}$ are lin. ind, i.e.
Assuming $Dx = 0$ for some $x \in \mathbb{R}^m$,

$$AD = I_m \implies \underbrace{Ax}_{0} = I_m x = x$$

$$\implies x = 0$$

This implies D has more rows than
columns, that is $n \geq m$

We conclude that $n = m$.

To see that $C = D$

$$CA = I_n \implies (CA)D = (I_n)D$$

$$\implies \underbrace{(CA)D}_{I_m} = D$$

$$\implies C = D$$

2.

$$AR_n = [a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ & & & & 1 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$= [Ae_n \ Ae_{n-1} \ \dots \ Ae_1]$$

$$= [a_n \ a_{n-1} \ \dots \ a_1]$$

$$R_n A = \begin{bmatrix} e_n^T \\ e_2^T \\ \vdots \\ e_1^T \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{bmatrix}$$

$$= \begin{bmatrix} e_n^T A \\ e_{n-1}^T A \\ \vdots \\ e_2^T A \\ e_1^T A \end{bmatrix} = \begin{bmatrix} \bar{a}_n \\ \bar{a}_{n-1} \\ \vdots \\ \bar{a}_2 \\ \bar{a}_1 \end{bmatrix}$$

3.

$$\begin{aligned} AP &= A [p_1 \ p_2 \ \dots \ p_n] \\ &= [Ap_1 \ Ap_2 \ \dots \ Ap_n] \\ &= [a_2 \ a_1 \ a_3 \ \dots \ a_n] \end{aligned}$$

$$\begin{aligned} Ap_1 &= a_2, & Ap_2 &= a_1, & Ap_3 &= a_3, \dots, & Ap_n &= a_n \\ \Rightarrow p_1 &= e_2, & \Rightarrow p_2 &= e_1, & \Rightarrow p_3 &= e_3, & \Rightarrow p_n &= e_n \end{aligned}$$

i.e

$$Ap_1 = a_1 p_{11} + a_2 p_{21} + \dots + a_n p_{n1} = a_2$$

where $p_1 = \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}$

~~$p_{11} = p_{31} = \dots = p_{n1} = 0$~~
 ~~$p_{21} = 1$ satisfies the vector equation~~

Check $PA = \begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \\ \bar{a}_n \end{bmatrix}$.

$$\begin{aligned} P &= [e_2 \ e_1 \ e_3 \ \dots \ e_n] \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_1^T \\ e_3^T \\ \vdots \\ e_n^T \end{bmatrix} \end{aligned}$$

$$PA = [e_2 \ e_1 \ e_3 \ \dots \ e_n] A$$

$$= \begin{bmatrix} e_2^T \\ e_1^T \\ e_3^T \\ \vdots \\ e_n^T \end{bmatrix} A = \begin{bmatrix} e_2^T A \\ e_1^T A \\ e_3^T A \\ \vdots \\ e_n^T A \end{bmatrix} = \begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \\ \bar{a}_3 \\ \vdots \\ \bar{a}_n \end{bmatrix}$$

$$2.2.3 \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}, \quad A^{-1} = \frac{1}{-40 + 35} \begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}$$

2.2.6 Solve the system

$$\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

A

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -14/5 \end{bmatrix} - \begin{bmatrix} 1 \\ -8/5 \end{bmatrix} = \begin{bmatrix} 1 \\ -6/5 \end{bmatrix}$$

2.2.21 If A is invertible,

A is row-equivalent to I_n .

~~Assume that $Ax = 0$, then $I_n x = 0 \Rightarrow x = 0$~~

Therefore $Ax = 0$ is satisfied only for $x = 0$.
The columns of A are lin. ind.

Conversely if A is not invertible

Reduced echelon form for A
has a zero row

$\Rightarrow Ax=0$ has free variables
or equivalently non-trivial solutions.

\Rightarrow Columns of A are lin. dep.

A is ~~not~~ invertible

The columns of A are lin. ind.

2.2.32

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array} \right]}_A \xrightarrow{(1) r_2 := r_2 - 4r_1} \xrightarrow{(2) r_3 := r_3 + 2r_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array} \right] \xrightarrow{(3) r_3 := r_3 - 2r_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array} \right]$$

0 row implies A^{-1} does not exist.

(6)

2.2.35. Since

$$AX = I_3 \text{ where } X = A^{-1}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

Third column of the inverse is the solution for

$$Ax_3 = e_3$$

$$\left[\begin{array}{ccc|c} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} (1) r_1 \leftrightarrow r_3 \\ (2) r_2 := r_2 - 2r_1 \\ (3) r_3 := r_3 + 2r_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} (4) r_3 := r_3 - r_2 \\ (5) r_2 := -r_2 \end{array}} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} (6) r_2 := r_2 - 2r_3 \\ (7) r_1 := r_1 - 4r_3 \end{array}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -15 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$\xrightarrow{(8) r_1 := r_1 - 3r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$x_3 = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$$

5

a. Let $Q = [q_1 \ q_2]$ with $q_1, q_2 \in \mathbb{R}^2$
 $Q^T = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}$ columns of the matrix

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$q_1^T q_1 = 1 \text{ means } \|q_1\|^2 = 1$$

$$q_1^T q_2 = 0 \text{ means } q_1 \perp q_2$$

$$q_2^T q_1 = 0 \text{ means } q_2 \perp q_1$$

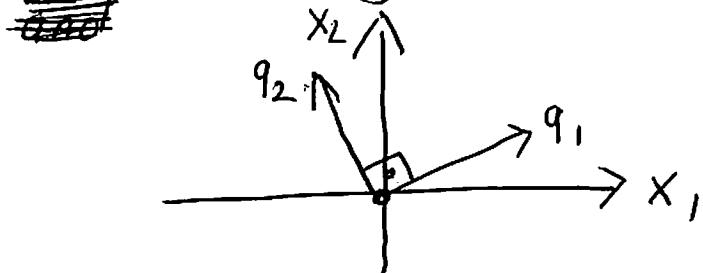
$$q_2^T q_2 = 1 \text{ means } \|q_2\|^2 = 1$$

NOTES: In general let $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$

$$a^T b = [a_1 \ a_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 \quad \left(\begin{array}{l} \text{Recall} \\ a^T b = 0 \text{ implies } a \perp b \end{array} \right)$$

$$a^T a = [a_1 \ a_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1^2 + a_2^2 = \|a\|^2 \quad \left(\begin{array}{l} \text{Recall that} \\ \|a\| = \sqrt{a_1^2 + a_2^2} \end{array} \right)$$

Geometrically Q has ^{mutually orthogonal} columns of length one.



(8)

For instance

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, q_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

are of length one
and mutually orthogonal.

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}, q_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

can also be the
columns of an orthogonal
matrix.

$$Q = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

b. Suppose

$$Qx = 0 \text{ for some } X$$

Multiply both sides by Q^T from the left.

$$Q^T Q x = Q^T 0 \implies x = 0$$

That is $Qx = 0$ holds only for $x = 0$.
columns of Q are lin. ind.

c. The system

$Qx = b$
is consistent, because

$$Q^T Q x = Q^T b \implies x = Q^T b \text{ is a solution.}$$

The solution is unique, since assuming
 $Qx = b$ and $Qy = b$

implies

But the column of Q are lin. ind. meaning
 $x - y = 0 \implies x = y$

(9)

d.

$$x = Q^T b = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix}$$

Verification

$$Qx = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} +2 \\ -1 \end{bmatrix} = b$$

2.3.1.

$$\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix} \text{ is invertible.}$$

Because the columns are not multiples of each other, they are lin. ind.

2.3.4

$$\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix} \text{ is not invertible.}$$

Because the second column is 0. Any set containing 0 is lin. dep.

2.3.18 Since $Cx = v$ is consistent for all v , the columns of C span \mathbb{R}^6 .

But from Theorem 8.1 we can deduce C is invertible. Since C is invertible, $Cx = v$ has a unique solution, namely $x = C^{-1}v$. (10)

7. (a) By inspection, it is clear that the columns of L are lin. ind.

Or you can reduce A into the reduced echelon form.

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{array} \right] \xrightarrow{(1) r_2 := r_2 + 2r_1} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 1 \end{array} \right]$$

$$\xrightarrow{(2) r_3 := r_3 - 3r_1} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$R = I_n$, so L is invertible

(b) Since L is invertible

$$(*) \boxed{\begin{array}{l} Ax = 0 \\ LU \\ \text{iff} \\ Ux = L^{-1}0 = 0 \end{array}}$$

If A is invertible, $Ax=0$ only for $x=0$

We deduce from (*) $Ux=0$ only for $x=0$.

The columns of U are lin. ind. meaning U is invertible.

If U is invertible, $Ux=0$ only for $x=0$

(*) implies $Ax=0$ only for $x=0$.

A is invertible.

(11)

(c) No, it is not. First two columns of U are multiples of each other. Since columns of U are lin. dep., U is not invertible.

Part (b) implies A is also not invertible.

(d) ~~If~~ Suppose an upper triangular matrix U has a zero along the diagonal.

$$U = \begin{bmatrix} * & * & - & - & * \\ * & * & - & - & * \\ 0 & * & & & \\ * & & & & \\ \vdots & & & & * \end{bmatrix}$$

But then $Ux=0$ has a free variable. U has lin. dep. columns and therefore is not invertible.

On the contrary if all the diagonal entries of U are non-zero,

$$U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & * \end{bmatrix}$$

$Ux=0$ has no free variables and the unique solution is $x=0$. Therefore the columns of U are lin. ind and therefore invertible. (1)

CONCLUSION: U is invertible if and only if all of its diagonal entries are non-zero

2.5.6.

L

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{array} \right]$$

U

$$\left[\begin{array}{cccc} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$x = \underbrace{\left[\begin{array}{c} 1 \\ -2 \\ -1 \\ 2 \end{array} \right]}_y$$

(1) Solve Ly = b

Augmented
matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 & -2 \\ 3 & -2 & 1 & 0 & -1 \\ -5 & 4 & -1 & 1 & 2 \end{array} \right] \quad \begin{array}{l} (1) r_2 := r_2 + 3r_1 \\ (2) r_3 := r_3 - 3r_1 \\ (3) r_4 := r_4 + 5r_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 4 & -1 & 1 & 7 \end{array} \right] \quad \begin{array}{l} (4) r_3 := r_3 + 2r_2 \\ (5) r_4 := r_4 - 4r_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 & 3 \end{array} \right] \quad \begin{array}{l} (6) r_4 := r_4 + r_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right], \quad y = \left[\begin{array}{c} 1 \\ 1 \\ -2 \\ 1 \end{array} \right]$$

(13)

(2) Solve $Ux = y$

$$\left[\begin{array}{ccccc} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 & 1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} (1) r_2 := r_2 - 2r_4 \\ (2) r_3 := r_3 / (-2) \end{array}}$$

$$\left[\begin{array}{ccccc} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} (3) r_2 := r_2 - 5r_3 \\ (4) r_1 := r_1 - 4r_3 \end{array}}$$

$$\left[\begin{array}{ccccc} 1 & 3 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} (5) r_1 := r_1 - r_2 \\ (6) r_2 := r_2 / 3 \end{array}}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$x = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

is the solution.

2. 5. 9.

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ -3 & -2 & 10 & 0 \\ 9 & -5 & 6 & 3 \end{array} \right] \xrightarrow{\begin{array}{l} (1) r_2 := r_2 + r_1 \\ (2) r_3 := r_3 - 3r_1 \end{array}} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & -3 & 12 & -1 \\ 0 & -2 & 0 & 3 \end{array} \right] \xrightarrow{E_1^{-1} E_2^{-1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & -2 & 0 & 3 \end{array} \right]$$

$$\xrightarrow{(3) r_2 := r_2 / -3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & -2 & 0 & 3 \end{array} \right] \xrightarrow{E_1^{-1} E_2^{-1} E_3^{-1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 \end{array} \right]$$

$$(4) \xrightarrow{r_3 := r_3 + 2r_2} \left[\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -1 & -3 & 0 \\ 0 & 0 & -8 & 3 & -2 & 1 \end{array} \right]$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & 0 \\ 3 & -2 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -8 \end{bmatrix}}_U$$

Note that LU-factorization is not unique

Another
LU-factorization $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}}_U$

2.5.17.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & 0 \\ 3 & -2 & 1 \end{bmatrix}}_L \quad \left| \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right. \quad \begin{array}{l} (1) r_2 := r_2 + r_1 \\ (2) r_3 := r_3 - 3r_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array} \right] \quad \begin{array}{l} (3) r_2 := r_2 / (-3) \\ (4) r_3 := r_3 + 2r_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array} \right] \quad \underbrace{\quad}_{L^{-1}}$$

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & -8 & 0 & 0 & 1 \end{array} \right] \xrightarrow{(1) r_3 := r_3 / (-8)} \left[\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/8 \end{array} \right] \xrightarrow{(2) r_2 := r_2 + 4r_3} \left[\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/8 \end{array} \right] \xrightarrow{(3) r_1 := r_1 - 2r_3} \left[\begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/8 \end{array} \right] \xrightarrow{(4) r_1 := r_1 + r_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/8 \end{array} \right] \xrightarrow{(5) r_1 := r_1 / 3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/8 \end{array} \right]$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1/3 & 1/3 & -1/2 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/3 & -1/2 & 0 \\ 0 & 0 & -1/8 & -1 & 2 & 1 \end{array} \right] \xrightarrow{U^{-1}} \left[\begin{array}{ccc|ccc} 1/3 & 1/3 & -1/2 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -1/3 & -1/2 & 0 \\ 0 & 0 & -1/8 & -1 & 2 & 1 \end{array} \right] = \left[\begin{array}{ccc|ccc} 1/6 & -5/18 & -1/12 & 1 & 0 & 0 \\ 1/6 & -4/3 & -1/2 & -1/3 & -1/2 & 0 \\ 1/8 & -1/4 & -1/8 & -1/4 & 1 & 1/2 \end{array} \right]$$

2.5.25. The equation

$$X(U D V^T) = I_n$$

is satisfied by

$$X = \boxed{V D^{-1} U^T}$$

$$\text{as } (\boxed{V D^{-1} U^T}) (V D V^T) = \boxed{V D^{-1} (U^T U)^{-1} D V^T} = \boxed{\sqrt{(D^{-1} D)^{-1}}_{\sqrt{n}}} = \boxed{\sqrt{V^T V}^{-1}_{\sqrt{n}}} = \boxed{I_n}$$

$$(16)$$

Note that inverse of the diagonal matrix is simply

$$D^{-1} = \begin{bmatrix} 1/\sigma_1 & & & 0 \\ & 1/\sigma_2 & \dots & \\ & & D & \dots & 1/\sigma_n \end{bmatrix}$$

NOTE:

If any of the diagonal entries of D (so called singular values) is zero, D^{-1} and A^{-1} do not exist.