

MATH 20F - WINTER 2008
 Homework 3, Solutions

4.1.2.

a. Let

$$u = \begin{bmatrix} x \\ y \end{bmatrix} \in W$$

meaning

$$xy \geq 0.$$

$$cu = \begin{bmatrix} cx \\ cy \end{bmatrix} \text{ with } (cx)(cy) = c^2 xy \geq 0$$

Therefore $cu \in W$ for all c .

b.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in W$$

But

$$v_1 + v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin W \quad (\text{since } 1)(-1) < 0)$$

4.1.8.

Set of polynomials in P_n
 satisfying $p(0) = 0$

$$\left\{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_1, a_2, \dots, a_n \in \mathbb{R} \right\}$$

$$\text{span} \{ x, x^2, \dots, x^n \}$$

Since the set is a span of vectors, it must be a subspace of P_n (1)

4.1.12.

$$W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Therefore W is 2-dimensional subspace of \mathbb{R}^4 .

4.1.20.

a. ① $C_a[a, b]$ is closed under addition

Let $f, g \in C[a, b]$

$$h(x) = f(x) + g(x)$$

The sum of two continuous functions is continuous.

$$\implies h \in C[a, b]$$

② $C[a, b]$ is closed under scalar multiplication

Let $f \in C[a, b]$

$$h(x) = c f(x)$$

Multiplying a continuous function with a scalar yields another continuous function. ②

$$\Rightarrow h \in [a, b]$$

$$③ f(x) = 0 \in [a, b]$$

In other words the constant function $f(x) = 0$ is continuous over $[a, b]$.

$$b. S = \{ f(x) \in [a, b] : f(a) = f(b) \}$$

① S is closed under addition

$$\text{Let } f, g \in S$$

$$h(x) = f(x) + g(x)$$

$$h(a) = f(a) + g(a)$$

$$h(b) = \overset{\text{and}}{f(b)} + g(b)$$

Therefore

$$h(a) = f(a) + g(a) = f(b) + g(b) = h(b)$$

implies $h \in S$

② S is closed under scalar multiplication

$$\text{Let } f \in S$$

$$\text{with } f(a) = f(b)$$

$$h(x) = cf(x) \in S$$

$$\text{since } h(a) = cf(a) = cf(b) = h(b)$$

③

③ S contains the function $f(x)=0$.

since $f(a) = f(b) = 0$

4.2.4.

$$\text{Null}(A) = \left\{ x \in \mathbb{R}^4 : \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} x = 0 \right\}$$

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

x_2, x_4 are free

$$x_3 = 0$$

$$x_1 - 6x_2 + 4x_3 = 0 \implies x_1 = 6x_2$$

A solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}$$

$$\text{Null}(A) = \left\{ \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(4)

4.2.8.

$$\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}$$

=

$$\left\{ \begin{bmatrix} \frac{s+2t+1}{5} \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{bmatrix} 1/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2/5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/5 \\ 0 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

The vector $\mathbf{0}$ is not in the set.

Therefore the set is not a vector space.

4.2.18.

Null(A) is a subspace of \mathbb{R}^3 .

since $x \in \text{Null}(A)$ iff $Ax = \mathbf{0}$

of cols of A = size of the vector x

Col(A) is a subspace of \mathbb{R}^4

$x \in \text{Col}(A) \implies x \in \text{span} \{ \underline{\mathbf{a}_1}, \underline{\mathbf{a}_2}, \underline{\mathbf{a}_3} \}$

Therefore $x \in \mathbb{R}^4$.

cols of A

(5)

4.2.32.

$$\begin{aligned} \text{Kernel}(T) &= \left\{ p \in P_2 : T(p) = 0 \right\} \\ &= \left\{ p \in P_2 : \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ a_0 + a_1 x + a_2 x^2 : \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ a_0 + a_1 x + a_2 x^2 : a_1, a_2 \in \mathbb{R} \right\} \\ &= \text{span} \{ x, x^2 \} \\ &\quad \begin{matrix} p_1 & p_2 \end{matrix} \end{aligned}$$

$$\begin{aligned} \text{Range}(T) &= \left\{ \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix} \in \mathbb{R}^2 : p(x) = a_0 + a_1 x + a_2 x^2 \right\} \\ &= \left\{ \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} : a_0 \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

$\text{Range}(T)$ is a one-dimensional subspace of \mathbb{R}^2 . It consists of all multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3.

(a)

$$B^{3 \times 3} = \left\{ a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

since $B^{3 \times 3}$ is spanned by a set of vectors, it is a subspace.

$$(b) B_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

from part (a) is a basis.

(i) It spans $B^{3 \times 3}$

(ii) It is lin. independent, i.e.

$$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & 0 \\ c_3 & c_1 & c_2 \\ 0 & c_3 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(7)

$$\implies c_1 = c_2 = c_3 = 0$$

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

is another basis, because it contains three vectors that are lin. independent.

$\dim(B^{3 \times 3}) = 3$, since any basis consists of 3 vectors.

4.

(a)

$$\begin{aligned} \text{Kernel}(T) &= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = 0 \right\} \\ &= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : \begin{bmatrix} a & b-c & 0 \\ a & a & b-c \\ 0 & a & a \end{bmatrix} = 0 \right\} \end{aligned}$$

$$\begin{bmatrix} a & b-c & 0 \\ a & a & b-c \\ 0 & a & a \end{bmatrix} = 0 \implies a=0 \\ b-c=0 \text{ or } b=c$$

$$\text{Kernel}(T) = \left\{ \begin{bmatrix} 0 \\ c \\ c \end{bmatrix} : c \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$(b) \quad \text{Range}(T) = \left\{ T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & b-c & 0 \\ a & a & b-c \\ 0 & a & a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

⑧

$$= \left\{ a \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

lin.
dependent
Basis

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

4.3.14

since A and B
are row-equivalent,

$$\text{Null}(A) = \{ x : Bx = 0 \}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_B \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_x = 0 \implies \begin{aligned} x_2, x_4 \text{ are free} \\ \underline{x_5 = 0} \quad (3\text{rd equation}) \\ 5x_3 - 7x_4 + 8x_5 = 0 \quad (2\text{nd equation}) \\ \implies \underline{x_3 = \frac{7}{5}x_4} \end{aligned}$$

Solution is
of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \frac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 4x_4 + \cancel{(5x_5)} = 0$$

$$x_1 = -2x_2 - 4x_4$$

$$\text{Null}(A) = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

(9)

$$= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

basis for Null(A)

$$\text{Col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

basis for Col(A)
(pivot columns, cols 1, 3, 5)
form a basis for Col(A)

4.3.20

Since v_1 can be written
as a l.m. combination of v_2 and v_3

$$H = \text{span} \{ v_1, v_2, v_3 \} = \text{span} \{ v_2, v_3 \}$$

$$= \text{span} \left\{ \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Basis since they
are l.m. independent

4.3.28.

$$H = \left\{ \cancel{v(t)} : Lv''(t) + Rv'(t) + (1/C)v(t) = 0 \right\}$$

$$= \left\{ e^{-bt}(c_1 + c_2 t) : \cancel{c_1, c_2 \in \mathbb{R}} \right\}$$

$$= \text{span} \{ e^{-bt}, te^{-bt} \}$$

⑯

$\{e^{-bt}, te^{-bt}\}$ is a basis for H .
 (Note that
 $b = \frac{R}{2L}$ is a constant)

4.3.32. Suppose $\{T(v_1), T(v_2), \dots, T(v_p)\}$
 is lin. dependent

$$c_1 T(v_1) + c_2 T(v_2) + \dots + c_p T(v_p) = 0$$

for some c_1, c_2, \dots, c_p not all zero

$$\implies \stackrel{\text{By linearity}}{T(c_1 v_1 + c_2 v_2 + \dots + c_p v_p)} = 0$$

Since T is one-to-one and $\stackrel{T(0)=0}{\text{(Because of linearity)}}$

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$$

for some c_1, c_2, \dots, c_p not all zero

That is $\{v_1, v_2, \dots, v_p\}$ is lin. dependent.

$$p_1(t) + p_2(t) = p_3(t)$$

Therefore $P = \text{span} \{p_1, p_2, p_3\} = \text{span} \{p_1, p_2\}$

Since $\{p_1, p_2\}$ are lin. independent
 $\{1+t, 1-t\}$ is a basis for P .

(11)

4.4.10 Let

$$v = \alpha_1 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Since

$v = [v]_E$: coordinates relative to the standard basis

$[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$: coordinates relative to B

$$[v]_E = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix} [v]_B$$

Change of ~~coordinates~~
matrix from B to
the standard basis

4.4.12.

$$\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_x = \alpha_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{[x]_B}$$

$$[x]_B = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}^{-1} \times = \frac{1}{2} \begin{bmatrix} 7 & -6 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

(12)

7. (a)

$$L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[L]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \neq$$

$$(b) [L_1]_{B_L} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}_{B_L} = (0) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (0) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$[L_2]_{B_L} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}_{B_L} = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (0) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[L_3]_{B_L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{B_L} = (1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (0) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$D = \left[[L_1]_{B_L}, [L_2]_{B_L}, [L_3]_{B_L} \right]$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(13)

(c)

$$[L]_{B_L} = D [L]_{C_L}$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} [L]_{C_L}$$

Solve the linear system by forming an augmented matrix

$$\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} -1 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} r_2 := r_2 + r_3 \\ r_1 := r_1 + r_2 \end{array}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} r_1 := -r_1 \\ r_2 := -r_2 \\ r_3 := -r_3 \end{array}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$[L]_{C_L} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$$4.5.4. \quad \left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$ — basis since the vectors are lin. indep.

A subspace of \mathbb{R}^4 of dimension 2

16

4.5.14.

A has 3 free variables

$$(x_2, x_5, x_6)$$

$$\dim \text{Null}(A) = 3$$

A has 3 pivot columns

(1st, 3rd and 4th cols)

$$\dim \text{Col}(A) = 3$$

4.6.4 A has 3 pivot columns

$$\text{rank}(A) = \dim \text{Col}(A) = 3$$

A has 3 nonpivot columns

$$\dim \text{Null}(A) = 3$$

Space
 $\text{Col}(A)$

$\text{row}(A)$

$$\begin{array}{c} \text{Basis} \\ \hline \begin{matrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{matrix} \end{array} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -5 \end{bmatrix} \right\}$$

pivot cols of A

$$\left\{ \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 1 & -1 & -1 & -2 \end{bmatrix} \right\}$$

nonzero rows of B
form a basis for $\text{row}(A)$

(15)

$$Bx = 0 \Rightarrow \left[\begin{array}{cccccc} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right] = 0$$

x_3, x_5, x_6 are free

$$\text{3rd row: } x_4 - x_5 - 2x_6 = 0 \Rightarrow x_4 = x_5 + 2x_6$$

$$\begin{aligned} \text{2nd row: } x_2 - x_3 + 3x_4 + 4x_5 - 3x_6 &= 0 \Rightarrow x_2 = x_3 - 3(x_5 + 2x_6) \\ &\quad + 4x_5 - 3x_6 \\ &= x_3 + x_5 - 9x_6 \end{aligned}$$

$$\text{1st row: } x_1 + x_2 - 3x_3 + 7x_4 + 9x_5 - 9x_6 = 0$$

$$\begin{aligned} &\Rightarrow x_1 = -x_2 + 3x_3 - 7x_4 - 9x_5 + 9x_6 \\ &= -(x_3 + x_5 - 9x_6) + 3x_3 + (x_5 + 2x_6) - 9x_5 + 9x_6 \\ &= 2x_3 - 9x_5 + 20x_6 \end{aligned}$$

Solution for $Bx = 0$ is of the form

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right] = \left[\begin{array}{c} 2x_3 - 9x_5 + 20x_6 \\ x_3 + x_5 - 9x_6 \\ x_5 + 2x_6 \\ x_6 \\ x_5 \\ x_6 \end{array} \right]$$

$$\text{Null}(A) = \{ x : Bx = 0 \}$$

$$= \left\{ x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 20 \\ -9 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} : x_3, x_5, x_6 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ -9 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Basis for Null(A)
(16)

4.6.10

$$\dim \text{Col}(A) + \overbrace{\dim \text{Null}(A)}^5 = \overbrace{7}^{\text{# of cols}}$$

$$\Rightarrow \dim \text{Col}(A) = 1$$

4.6.14

In both cases $\dim \text{Row}(A) = 3$ at most

Recall that

$$\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Row}(A)$$

If A is 4×3

$$\dim \text{Row}(A) = \underline{\dim \text{Col}(A) \leq 3}$$

Since A has 3 columns.

If A is 3×4

$$\dim \text{Col}(A) = \underline{\dim \text{Row}(A) \leq 3}$$

since A has 3 rows

4.6.20

To restate the problem, is

$Ax = b$ consistent for some b

if A is 6×8 and has two free variables.

$$\dim \text{Null}(A) = \text{number of free variables} \\ = 2$$

$$\dim \text{Col}(A) + \frac{\dim \text{Null}(A)}{2} = \frac{\text{number of cols}}{2} \\ = 8$$

$$\implies \dim \text{Col}(A) = 6$$

6 of the columns of A are lin. independent and must span \mathbb{R}^6 .

$$\begin{matrix} Ax = b \\ 6 \times 8 \end{matrix} \quad \begin{matrix} \text{in } \mathbb{R}^6 \\ \text{for all } b \in \mathbb{R}^6 \end{matrix} \quad \begin{matrix} \text{is consistent} \\ \text{for all } b \in \mathbb{R}^6. \end{matrix}$$

10. (a) A is not invertible, because Σ has ~~two~~ diagonal entries, (~~two~~ of the singular values are zero.) equal to zero.

For instance choose

$$V = \begin{bmatrix} 0.4\sqrt{2} \\ 1/\sqrt{2} \\ 0.3\sqrt{2} \end{bmatrix} \quad \begin{matrix} \text{transpose of} \\ \text{the last row of } V^T \end{matrix}$$

$$A \mathbf{v} = U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.4\sqrt{2} & -1/\sqrt{2} & 0.3\sqrt{2} \\ 0.6 & 0 & -0.8 \\ 0.4\sqrt{2} & 1/\sqrt{2} & 0.3\sqrt{2} \end{bmatrix} \begin{bmatrix} 0.4\sqrt{2} \\ 1/\sqrt{2} \\ 0.3\sqrt{2} \end{bmatrix}$$

$\mathbf{v}_1^T \quad \mathbf{v}_2^T \quad \mathbf{v}_3^T$

Since \mathbf{v} is orthogonal

$$\mathbf{v}_1^T \mathbf{v}_3 = 0, \quad \mathbf{v}_2^T \mathbf{v}_3 = 0, \quad \mathbf{v}_3^T \mathbf{v}_3 = 1.$$

$$\begin{aligned} A\mathbf{v} &= U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= U \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \end{aligned}$$

$A\mathbf{v} = \mathbf{0}$ for some nonzero \mathbf{v} .

The cols of A are lin. dependent
 A is not invertible.

(b) Notice that

$$\begin{aligned} A\mathbf{v}_2 &= U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix} \mathbf{v}_2 \\ &= U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = U \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \end{aligned}$$

In the previous part we showed that
 $A\mathbf{v}_3 = \mathbf{0}$

(19)

On the other hand

$$Av_1 = U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} v_1$$

$$= U \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = U \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

v_1 (first column
of U)

For Any vector $v \in \text{span}\{v_2, v_3\}$

$$Av = A(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) = \alpha_1 \underbrace{Av_1}_0 + \alpha_2 \underbrace{Av_2}_0 + \alpha_3 \underbrace{Av_3}_0 = 0$$

On the other hand if $v \notin \text{span}\{v_2, v_3\}$

$$v = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \text{ with } \beta_i \neq 0$$

$$Av = A(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3) = \beta_1 Av_1 \neq 0$$

Therefore

$$\text{Null}(A) = \text{span}\{v_2, v_3\}$$

$\{v_2, v_3\}$ is a basis for $\text{Null}(A)$,
since it is a lin. independent set.

$$(c) \quad U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ u_1 & u_2 & u_3 \end{bmatrix}$$

$$A = [u_1 \ u_2 \ u_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

$$= [2u_1 \ 0 \ 0] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

From part (c), the only way $Av \neq 0$ if v has a component along v_1 that is

$$v = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \text{ with } \beta_1 \neq 0$$

Choose such a v

$$Av = [2u_1 \ 0 \ 0] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} (\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3)$$

$$\leq [2u_1 \ 0 \ 0] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \beta_1 v_1$$

$$= [2u_1 \ 0 \ 0] \begin{bmatrix} \beta_1 \\ 0 \\ 0 \end{bmatrix} = 2u_1$$

~~$\text{Then } \text{Col}(A) = \left\{ \frac{Av}{\alpha_{u_1}} : v \in \mathbb{R}^3 \right\}$~~

~~$= \text{span}(u_1)$~~ basis for $\text{Col}(A)$

(21)

$$(d) \quad \text{rank}(A) = \dim(\text{col}(A)) \\ = 1$$

Notice that $\text{rank}(A)$ is equal to the number of nonzero diagonal entries of Σ .