

Name:	PID:	
Discussion Section		
No:	Time:	TA:

Midterm 2, Math 20F - Winter 2008

Duration: 100 minutes

This is an open book exam. Calculators are not allowed.

To get full credit you should support your answers.

1. Given

$$A = \begin{bmatrix} -1 & 3 & -2 \\ -3 & 5 & 4 \\ 3 & -9 & 2 \end{bmatrix}$$

(a) (1 point) Determine whether A is invertible or not? If it is invertible, find its inverse.

Answer for the other type

$$A^{-1} = \begin{bmatrix} -13/9 & -2/3 & -8/9 \\ -8/9 & -1/3 & -7/9 \\ 2/3 & 0 & -1/3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 5 & 4 & 0 & 1 & 0 \\ 3 & -9 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{(1) r_2 := r_2 - 3r_1 \\ (2) r_3 := r_3 + 3r_1}} \left[\begin{array}{ccc|ccc} -1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -4 & 10 & -3 & 1 & 0 \\ 0 & 0 & -4 & 3 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\substack{(3) r_3 := r_3 / (-4) \\ (4) r_2 := r_2 / (-4) \\ (5) r_1 := -r_1}} \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & -1 & 0 & 0 \\ 0 & 1 & -10/4 & 3/4 & -1/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 & -1/4 \end{array} \right] \xrightarrow{\substack{(6) r_2 := r_2 + 10/4 r_3 \\ (7) r_1 := r_1 - 2r_3 \\ (8) r_1 := r_1 + 3r_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -23/8 & -3/4 & -11/8 \\ 0 & 1 & 0 & -9/8 & -1/4 & -5/8 \\ 0 & 0 & 1 & -3/4 & 0 & -1/4 \end{array} \right]$$

A^{-1} inverse

(b) (1 point) Based on your answer to part (a), indicate whether each of the following is true or false with a brief explanation.

- (i) The null space of A consists of the zero vector only. True
 Since A is invertible, the cols of A are linearly independent
 $\implies Ax = 0$ only for $x = 0$.
 $\implies \text{Null}(A) = \{0\}$
- (ii) There exists a $b \in \mathbb{R}^3$ such that the system $Ax = b$ is inconsistent. False

Since A is invertible,
 the cols of A span \mathbb{R}^3
 i.e. $b \in \text{Col}(A)$ for all b
 $\implies Ax = b$ is consistent
 for all b .

#	Score
1 (4 points)	
2 (4 points)	
3 (4 points)	
4 (4 points)	
5 (4 points)	
Total (20 points)	

(c) (2 points) Find an LU-factorization for A. Is this LU-factorization unique? If it is, explain why. If not, give another LU-factorization for A.

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 5 & 4 & 0 & 1 & 0 \\ 3 & -9 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{(1) r_2 := r_2 - 3r_1 \\ (2) r_3 := r_3 + 3r_1}} \left[\begin{array}{ccc|ccc} -1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -4 & 10 & 3 & 1 & 0 \\ 0 & 0 & -4 & -3 & 0 & 1 \end{array} \right] E_2^{-1} E_1^{-1}$$

where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

elementary matrix corresponding to $r_2 := r_2 - 3r_1$, elementary matrix corresponding to $r_3 := r_3 + 3r_1$

LU factorization $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} -1 & 3 & -2 \\ 0 & -4 & 10 \\ 0 & 0 & -4 \end{bmatrix}}_U$

LU factorization is not unique; $A = 3 \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.
 for instance multiply L with 3 and divide U by 3, you obtain another LU factorization $\left(\frac{1}{3} \begin{bmatrix} -1 & 3 & -2 \\ 0 & -4 & 10 \\ 0 & 0 & -4 \end{bmatrix} \right)$

Another answer for the other type $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 0 & -3 & 7 \\ 0 & 0 & -3 \end{bmatrix}$

2. Let A be a 3×5 matrix. Suppose that the system $Ax = b$ is consistent for all $b \in \mathbb{R}^3$. In each part justify your answer.

(a) (1 point) Determine the rank of A.

$Ax = b$ is consistent for all $b \in \mathbb{R}^3$
 $\implies \text{Col}(A) = \mathbb{R}^3$
 $\implies \dim \text{Col}(A) = \text{rank}(A) = 3$

Answer for the other type $\text{rank}(A) = 4$

(b) (1 point) Determine the dimension of the null space of A.

$\dim \text{Col}(A) + \dim \text{Null}(A) = \# \text{ of columns}$
 $3 + \dim \text{Null}(A) = 5$
 $\implies \dim \text{Null}(A) = 2$

Answer for the other type $\dim \text{Null}(A) = 2$

(c) (2 points) Is the system $A^T x = c$ consistent for all c ?

No, it is not.

A^T is 5×3

3 column vectors cannot span \mathbb{R}^5
 \implies There exists a c such that $c \notin \text{Col}(A^T)$
 \implies There exists a c such that $A^T x = c$ is inconsistent

Answer for the other type $A^T x = c$ is not consistent for all c

3. A QR-factorization for the matrix A is given by

$$A = \overbrace{\begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.8 & -0.6 \\ 0 & 0 & -0.6 & 0.8 \\ 0.6 & 0.8 & 0 & 0 \end{bmatrix}}^Q \overbrace{\begin{bmatrix} -3 & 10 & 8 & -1 \\ 0 & 3 & -4 & 3 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & 0 & 2 \end{bmatrix}}^R$$

where $Q^T Q = I$.

(a) (2 points) Without constructing the matrix A solve the linear system

$$Ax = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

by exploiting the QR-factorization.

① $QRx = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \Rightarrow Rx = Q^T \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$

② Solve $\begin{bmatrix} -3 & 10 & 8 & -1 \\ 0 & 3 & -4 & 3 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ -14 \\ 2 \end{bmatrix}$

~~$\begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & -0.8 & -0.6 \\ 0 & 0 & -0.6 & 0.8 \\ 0.6 & 0.8 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ -14 \\ 2 \end{bmatrix}$~~

$= \begin{bmatrix} 0.8 & 0 & 0 & 0.6 \\ -0.6 & 0 & 0 & 0.8 \\ 0 & -0.8 & -0.6 & 0 \\ 0 & -0.6 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ -14 \\ 2 \end{bmatrix}$

(b) (2 points) Without constructing A decide which of the matrices Q , R and A are invertible? Explain your reasoning.

$Q^T Q = I \Rightarrow Q$ is invertible

R is invertible, since its cols are l.n. ind.

~~$A^{-1} = (R^{-1} Q^T)$~~

$(R^{-1} Q^T) \underbrace{Q}_{A^{-1}} \underbrace{R}_A = I_n$

Therefore A is also invertible.

Answer for the other type $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

4. The set

$$\mathbb{S}^{2 \times 2} = \left\{ \begin{bmatrix} b & a \\ a & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

is a subset of 2×2 symmetric matrices. Specifically if $A \in \mathbb{S}^{2 \times 2}$, it satisfies the property $A^T = A$. Let $T: \mathbb{R}^2 \rightarrow \mathbb{S}^{2 \times 2}$ be the linear transformation defined as

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & a+b \\ a+b & 0 \end{bmatrix}.$$

(a) (1 point) Write down a symmetric matrix A satisfying $A^T = A$ that does not belong to $\mathbb{S}^{2 \times 2}$.

Any symmetric matrix with different entries along the diagonal

e.g. $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

Answer for the other type is the same.

(b) (1 point) Show that $\mathbb{S}^{2 \times 2}$ is a vector space.

Answer for the other type is the same.

$$\mathbb{S}^{2 \times 2} = \left\{ b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Any set spanned by a set of vectors is a vector space.

(c) (1 point) Give a basis B for $\mathbb{S}^{2 \times 2}$. Find the coordinate vector $[A]_B$ of the matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ relative to B . Therefore $\mathbb{S}^{2 \times 2}$ is a vector space.

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

relative to B .

Answer for the other type
The basis is the same.
 $[A]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis
(i) The set is linearly independent, as the matrices are not multiples of each other.
(ii) The set spans $\mathbb{S}^{2 \times 2}$ as shown in (b)

(d) (1 point) Find a basis for the Kernel of T .

$$\text{Kernel}(T) = \left\{ v : T(v) = 0 \right\}$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow [A]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Answer for the other type
 $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Kernel}(T)$

$$= \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : \begin{bmatrix} 0 & a+b \\ a+b & 0 \end{bmatrix} = 0 \right\}$$

implies $a = -b$

$$= \left\{ \begin{bmatrix} -b \\ b \end{bmatrix} : b \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$B = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Kernel}(T)$.

5. Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) Answer for the other type:

satisfying the property $A^T = A$. Symmetric matrices possess some remarkable properties. One such property is that for an $n \times n$ symmetric matrix all vectors $v \in \mathbb{R}^n$ can be written of the form $v = v_n + v_c$ where v_n belongs to the null space and v_c belongs to the column space of the matrix. Below you will verify this property specifically for A.

$$B_c = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$$

$$B_n = \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$B_r = \left\{ \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \right\}$$

(3 points) Find bases B_c , B_n and B_r for the column, null and row space of A, respectively.

Row-reduce A into an echelon form.

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 := r_2 - \frac{3}{2}r_1} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1/2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_3 := r_3 - 2r_2} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\text{Col}(A)$ the pivot columns form a basis: $B_c = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$

For $\text{Null}(A)$ we need to solve $Ux = 0$. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} 1/2 x_2 + x_3 = 0 \Rightarrow x_2 = -2x_3 \\ 2x_1 + 3x_2 = 0 \Rightarrow x_1 = -3/2 x_2 = 3x_3 \end{cases}$
 $\text{Null}(A) = \text{span} \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$: $B_n = \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$

For $\text{Row}(A)$ the non-zero rows of U form a basis: $B_r = \left\{ \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 & 1 \end{bmatrix} \right\}$

(b) (1 point) Verify that the union of the bases B_c and B_n forms a basis for \mathbb{R}^3 .

The union of B_c and B_n

$$B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a linearly independent set.}$$

Row-reduction

$$\text{i.e. } \begin{bmatrix} 2 & 3 & -3 \\ 3 & 5 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_2 := r_2 - \frac{3}{2}r_1} \begin{bmatrix} 2 & 3 & -3 \\ 0 & 1/2 & 5/2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 := r_3 - 2r_2} \begin{bmatrix} 2 & 3 & -3 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & -6 \end{bmatrix}$$

indicates that B is linearly independent.

Since three lin. independent vectors in \mathbb{R}^3 form a basis for \mathbb{R}^3 , B must be a basis.

Answer for the other type: Similar, you need to show that $B_c \cup B_n$ is lin. ind.