Name: **PID:**
Discussion Section - No: **Time:**

Final, Math 20F - Lecture B (Spring 2007)

Duration: 3 hours
This is a closed-book exam. Calculators are not allowed. You can use one page of notes.
To get full credit you should support your answers unless otherwise is stated.
There are eight questions in this exam.

1.

a) (4 points) An LU-factorization for a 4×4 matrix A is given as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Evaluate the determinant of A. Is A invertible?

Question#	Score
1 (7pts)	
2 (7pts)	
3 (7pts)	
4 (7pts)	
5 (8pts)	
6 (6pts)	
7 (7pts)	
8 (6pts)	
Total (55pts)	

b) (3 points) Evaluate the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

2. (Each part is 1 point) Determine whether each of the following statements is true or false. For each part circle either **T**(true) or **F**(false). You do not need to justify your answers.

(i) An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvectors that are mutually orthogonal. __**T**__ __**F**__

(ii) The orthogonal projection of a vector v onto a subspace $\mathcal{W}$ yields the component $\hat{v}$ of v contained on the subspace $\mathcal{W}$ such that $(v - \hat{v})$ is orthogonal to $\hat{v}$. __**T**__ __**F**__

(iii) The set of polynomials is an infinite dimensional vector space. __**T**__ __**F**__

(iv) Given an $n \times n$ matrix A, if the linear system $Ax = b$ is inconsistent for some b, then A is not invertible. __**T**__ __**F**__

(v) The set $\{c_1c_2 + c_1x + c_2x^2 : \; c_1, c_2 \in \mathbb{R}^2\}$ is a two dimensional subspace of the vector space of polynomials of degree at most two $\mathbb{P}_2 = \{c_0 + c_1x + c_2x^2 : c_0, c_1, c_2 \in \mathbb{R}\}$.
__**T**__ __**F**__

(vi) If the rank of a 5×7 matrix A is 4, then the null space of A is a 1-dimensional subspace of $\mathbb{R}^5$. __**T**__ __**F**__

(vii) Let v be a vector in an n-dimensional vector space $\mathcal{V}$ spanned by the basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$. If $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n = \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_n b_n$ for some $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n \in \mathbb{R}$, then $\alpha_1 = \beta_1, \; \alpha_2 = \beta_2, \; \ldots, \; \alpha_n = \beta_n$, that is there is a unique way v can be written as a linear combination of $b_1, b_2, \ldots, b_n$. __**T**__ __**F**__

3. Suppose a QR-factorization of A is given by

$$\overbrace{\begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ 4 & 2 & 2 \end{bmatrix}}^{A} = \overbrace{\left(\frac{1}{5}\begin{bmatrix} 3 & -4 & 0 \\ 0 & 0 & 5 \\ 4 & 3 & 0 \end{bmatrix}\right)}^{Q} \overbrace{\begin{bmatrix} 5 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}}^{R}$$

where Q is an orthogonal matrix (that is $Q^TQ = I$, therefore Q^T is the inverse of Q).

a) (4 points) Using the QR-factorization above find the inverse of A.

b) (3 points) Using the QR-factorization solve the linear system

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

a) (3 points) Find a basis for the column space of A. What is the rank of A? Is A invertible?

b) (4 points) Find all least squares solutions $\hat{x}$ satisfying

$$\|b - A\hat{x}\| \leq \|b - Ax\| \text{ for all } x \in \mathbb{R}^3.$$

5. Suppose A is a 3×3 matrix with the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$ and the eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

corresponding to λ_1, λ_2 and λ_3, respectively. Consider also the dynamical system

$$x_k = Ax_{k-1}$$

with the initial condition

$$x_0 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

a) (1 point) Write x_0 as a linear combination of v_1, v_2 and v_3, that is determine the scalars α_1, α_2 and α_3 such that

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3.$$

b) (4 points) Using the expression for x_0 as a linear combination of the eigenvectors you determined in part **a)** find $x_3 = A^3 x_0$.

c) (3 points) What are the eigenvalues and the corresponding eigenvectors of $A - 2I$.

6. Let $\mathbb{P}_2 = \{c_1 + c_2x + c_3x^2 \;:\; c_1, c_2, c_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most two. Define the linear transformation $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ as

$$T(c_1 + c_2x + c_3x^2) = \begin{bmatrix} c_1 \\ c_2 + c_3 \end{bmatrix}.$$

a) (3 points) Find a basis for the kernel of T. Is T one-to-one?

b) (3 points) Find a basis for the range of T. Is T onto $\mathbb{R}^2$?

7. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

a) (2 points) Is the set $\{v_1, v_2, v_3\}$ linearly independent?

b) (2 points) Is the set $\{v_1, v_2, v_3\}$ orthogonal?

c) (3 points) Find an orthogonal basis for the subspace $\mathcal{W} = \text{span}\{v_1, v_2, v_3\}$.

8. (6 points) Find the eigenvalues and the corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

Indicate which eigenvector corresponds to which eigenvalue. Is A diagonalizable?