Midterm 1, Math 20F - Lecture B (Spring 2007)

1. Consider the system of linear equations

$$2x_1 + 4x_2 - x_3 = 2$$
$$x_1 + 2x_2 + hx_3 = b$$

where h and b are scalars.

a) (2.5 points) Explain why the system cannot have a unique solution.

Solution:
The augmented matrix associated with this system can have at most two leading entries, since it has two rows and each row can contain at most one leading entry. This means that the augmented matrix has at most two pivot columns. Therefore the system must have at least one free variable. The system can be inconsistent, if the last column of the augmented matrix is the pivot column. Otherwise, if the system is consistent, the system has a free variable implying the system has infinitely many solutions.

b) (2.5 points) For what values of h and b is the system inconsistent?

Solution:
The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & -1 & 2 \\ 1 & 2 & h & b \end{bmatrix}.$$

Adding $-1/2$ times the first row to the second one yields

$$\begin{bmatrix} 2 & 4 & -1 & 2 \\ 0 & 0 & h+0.5 & b-1 \end{bmatrix}$$

The system is inconsistent if the last column is a pivot column. This is the case whenever

$$h + 0.5 = 0 \text{ and } b - 1 \neq 0,$$

or equivalently the system is inconsistent whenever

$$h = -0.5 \text{ and } b \neq -1.$$

2. The solution set of the homogeneous system $Ax = 0$ where A is a 3 by 3 matrix (corresponding to a system of three equations in three unknowns) is given by

$$\text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

Let

$$p = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

be a particular solution satisfying $Ax = b$ for some given vector b of size 3.

a) (2 points) Give a solution of the system $Ax = b$ not equal to the particular solution p.

Solution:
Any solution for the system $Ax = b$ is in the form

$$p + v_h$$

where p is the particular solution and v_h is a solution for the homogeneous system $Ax = 0$. You can pick v_h any vector in

$$\text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

In particular

$$v_h = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a solution for the homogenous system and

$$p + v_h = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

must be a solution for the system $Ax = b$.

b) (3 points) Based on the solution for the homogeneous system given above provide a reduced echelon matrix that is possibly row-equivalent to A.

Solution:

The span of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the solution set for $Ax = 0$ meaning for any x_1 and x_2.

$$A\left(x_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = A\left(\begin{bmatrix} -x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix}\right) = 0.$$

Now let R be the reduced echelon form for A. As the row-reduction operations preserve the solution

$$R\left(\begin{bmatrix} -x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix}\right) = 0. \tag{1}$$

Since x_1 and x_2 are free variables, one possibility is that only the first column of R is the pivot column. For instance one possibility is that its last two rows are zero and its first row is non-zero in the form $[1 \;\; r_1 \;\; r_2]$, that is

$$R = \begin{bmatrix} 1 & r_1 & r_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But then the uppermost linear equation in (1)

$$-x_1 - x_2 + r_1 x_1 + r_2 x_2 = (r_1 - 1)x_1 + (r_2 - 1)x_2 = 0$$

holds for all x_1 and x_2 only if $(r_1 - 1) = 0$ and $(r_2 - 1) = 0$. That is

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Let

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ -1 \\ h \end{bmatrix}$$

where h is a scalar.

a) (2 points) Do v_1 and v_2 span $\mathbb{R}^3$? (or equivalently can any vector in $\mathbb{R}^3$ be expressed as a linear combination of v_1 and v_2?) If it spans, explain why. If not, provide a vector of size 3 that cannot be expressed as a linear combination of v_1 and v_2.

Solution:
The span of two vectors in $\mathbb{R}^3$ is a line (if they are linearly dependent) or a plane (if they are linearly independent). No two vectors in $\mathbb{R}^3$ can span $\mathbb{R}^3$. To formally show this the span of v_1 and v_2 consist of all those b such that

$$x_1 v_1 + x_2 v_2 = b \tag{2}$$

for some scalars x_1 and x_2 or

$$[v_1 \;\; v_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Vx = b.$$

This is a linear system with three equations in two unknowns. Then at least one of the rows of V does not contain any leading entry. By Theorem 4 (which states that the columns of a m by n matrix A span $\mathbb{R}^m$ if and only if each row contains a leading entry) in your book v_1 and v_2 cannot span $\mathbb{R}^3$.

For instance the vector

$$b = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

does not lie in the span of v_1 and v_2. You can set up a system and verify that it is inconsistent. A more practical way is, since the second entry of the vector above is -1, we need to set $x_1 = 1$ in (2) (as the second entry of v_2 is zero). But then in order to obtain the first entry of b as -1, we need to have $x_2 = 3$. But the third entry of $v_1 + 3v_2$ is 6. Therefore b is not in the span of v_1 and v_2.

b) (3 points) For what values of h do the vectors v_1, v_2 and v_3 span $\mathbb{R}^3$?

Solution:
These three vectors span $\mathbb{R}^3$ if and only if they are linearly independent or equivalently the homogeneous system

$$[v_1 \ \ v_2 \ \ v_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

has no non-trivial solution which is the case whenever the system has no free variable. To see when the system above has no free variables, we need to reduce the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 2 & h \end{bmatrix}$$

into an echelon form. Swapping first row with the second one and adding two times the first row to the second row after the swap operation gives

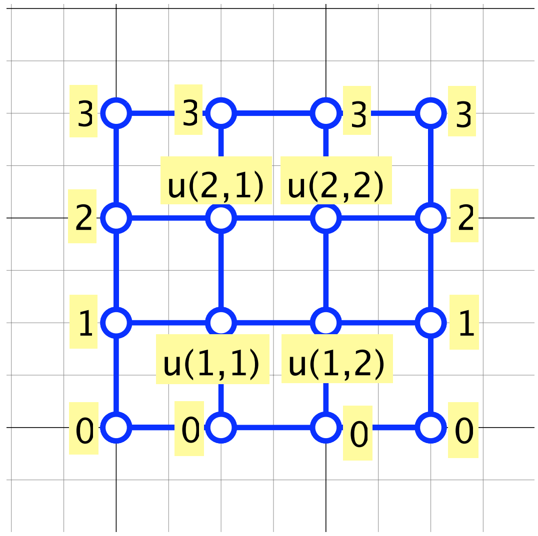
$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & h \end{bmatrix}.$$

Now adding twice the second row to the third yields an echelon matrix.

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & h-4 \end{bmatrix}$$

The system has no free variables, therefore v_1, v_2, v_3 are linearly independent and span $\mathbb{R}^3$, whenever the last column is a pivot column, that is whenever $h \neq 4$.

4. Consider the 4 by 4 grid illustrated below that will be used to approximate the density of a fluid inside a container at various positions.



For the nodes on the boundary of the container the density values are given on the figure. At any node inside the container the density is approximated by the average of the surrounding nodes.

a) (2.5 points) Set up a system of 4 linear equations (one for each of the nodes not on the boundary) with unknowns $u(1,1)$ (the density at $(1,1)$), $u(1,2)$ (the density at $(1,2)$), $u(2,1)$ (the density at $(2,1)$) and $u(2,2)$ (the density at $(2,2)$).

Solution:
The density $u(1,1)$ is the average of the densities $u(2,1)$, $u(1,2)$, 0 and 1.

$$u(1,1) = \frac{u(2,1)+u(1,2)+0+1}{4} \Longrightarrow 4u(1,1) - u(2,1) - u(1,2) = 1.$$

The density $u(1,2)$ is the average of the densities $u(1,1)$, $u(2,2)$, 0 and 1

$$u(1,2) = \frac{u(1,1)+u(2,2)+0+1}{4} \Longrightarrow 4u(1,2) - u(1,1) - u(2,2) = 1.$$

The density $u(2,1)$ is the average of the densities $u(1,1)$, $u(2,2)$, 2 and 3

$$u(2,1) = \frac{u(1,1)+u(2,2)+2+3}{4} \Longrightarrow 4u(2,1) - u(1,1) - u(2,2) = 5.$$

The density $u(2,2)$ is the average of the densities $u(2,1)$, $u(1,2)$, 2 and 3.

$$u(2,2) = \frac{u(2,1)+u(1,2)+2+3}{4} \Longrightarrow 4u(2,2) - u(2,1) - u(1,2) = 5.$$

b) (2.5 points) Solve the system of linear equations in part **a)** by reducing it to the reduced echelon form using the row operations. (Hint: sometimes performing the row-swap operation, when it is not essential, may avoid the entries of the matrix becoming very small or large and simplify the calculations.)

Solution:

The system of linear equations can be expressed as the following matrix equation.

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u(1,1) \\ u(1,2) \\ u(2,1) \\ u(2,2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 5 \end{bmatrix}$$

We apply the row operations to the augmented matrix for the system to reduce it to the echelon form. (The order of the operations matter; for instance when making the entries on the first column zero, first and second rows are swapped, after the swap operation four times the new first row is added to the new second row.)

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 1 \\ -1 & 4 & 0 & -1 & 1 \\ -1 & 0 & 4 & -1 & 5 \\ 0 & -1 & -1 & 4 & 5 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_2,\ r_2 := r_2 + 4r_1,\ r_3 := r_3 - r_1]{} \begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & 15 & -1 & -4 & 5 \\ 0 & -4 & 4 & 0 & 4 \\ 0 & -1 & -1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & 15 & -1 & -4 & 5 \\ 0 & -4 & 4 & 0 & 4 \\ 0 & -1 & -1 & 4 & 5 \end{bmatrix} \xrightarrow[r_2 \leftrightarrow r_4,\ r_3 := r_3 - 4r_2,\ r_4 := r_4 + 15r_2]{} \begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & -1 & -1 & 4 & 5 \\ 0 & 0 & 8 & -16 & -16 \\ 0 & 0 & -16 & 56 & 80 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & -1 & -1 & 4 & 5 \\ 0 & 0 & 8 & -16 & -16 \\ 0 & 0 & -16 & 56 & 80 \end{bmatrix} \xrightarrow[r_3 := \frac{r_3}{8},\ r_4 := \frac{r_4}{8},\ r_4 := r_4 + 2r_3\ r_4 := \frac{r_4}{3}]{} \begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & -1 & -1 & 4 & 5 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

To obtain the echelon form we start from the fourth column and move towards the first column making all the entries equal to zero except those for which the row and column numbers are the same, which we make equal to one.

$$\begin{bmatrix} -1 & 4 & 0 & -1 & 1 \\ 0 & -1 & -1 & 4 & 5 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 := r_3 + 2r_4,\ r_2 := r_2 - 4r_4,\ r_1 := r_4 + r_1]{} \begin{bmatrix} -1 & 4 & 0 & 0 & 3 \\ 0 & -1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & 0 & 0 & 3 \\ 0 & -1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 := r_2 + r_3,\ r_1 := r_1 + 4r_2]{} \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 := -r_2\ r_3 := -r_3]{} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Therefore $u(2,2) = u(2,1) = 2$ and $u(1,1) = u(1,2) = 1$.