**Name:** **PID:**
**Discussion Section - No:** **Time:**

# Midterm 2, Math 20F - Lecture B (Spring 2007)

*Duration: 50 minutes*
*This is a closed-book exam. Calculators are not allowed. You can use one page of notes.*
*To get full credit you should support your answers unless otherwise is stated.*
*There are four questions in this exam.*

**1.** An $n \times n$ matrix $A$ is called skew-symmetric if $A^T = -A$. Specifically the set of $2 \times 2$ skew-symmetric matrices is given by

$$S^{2\times 2} = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \; : \; a \in \mathbb{R} \right\}.$$

Let $T \; : \; L^{2\times 2} \to S^{2\times 2}$ be the transformation from the $2 \times 2$ lower triangular matrices onto the $2 \times 2$ skew-symmetric matrices defined as

$$T\left( \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}.$$

where $a$, $b$, $c$ are real numbers.

**a) (1 point)** Show that $S^{2\times 2}$ is a one-dimensional subspace of $2 \times 2$ matrices. Find a basis for this subspace.

**b) (2 points)** Show that $T$ is a *linear* transformation.

| # | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
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| Total | |

**c) (2 points)** Find a basis for the kernel of $T$.

**2. (Each part is 1 point)** Determine whether each of the following statements is true or false. For each part circle either **T**(true) or **F**(false). You do not need to justify your answers.

**(i)** The matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is invertible. **T** **F**

**(ii)** Any $n \times n$ matrix $A$ has an $LU$ factorization, that is there exist a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$. **T** **F**

**(iii)** The column space of an $n \times m$ matrix $A$ is preserved when an elementary row operation is applied. That is if $E$ is an $n \times n$ elementary matrix, the column spaces of $EA$ and $A$ are the same. **T** **F**

**(iv)** If the kernel of a linear transformation from a vector space to another vector space is $\{0\}$, then it is one-to-one. **T** **F**

**(v)** The set $\{1, 1+x, 2+x^2, x+x^2\}$ is a basis for $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 \ : \ a_0, a_1, a_2 \in \mathbb{R}\}$, the vector space of polynomials of degree at most two. **T** **F**

**3.** Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

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**a) (2 points)** Find an $LU$ factorization for $A$.

**b) (3 points)** Find the inverse of $A$.

**4.** The singular matrices can also be characterized in terms of their $LU$ factorizations.

**a) (3 points)** Suppose $A = LU$ is an $LU$ factorization for an $n \times n$ matrix $A$. Given that $L$ is invertible (the procedure based on the elementary matrices to find an $LU$ factorization always yields an invertible lower triangular matrix) prove that $A$ is singular if and only if $U$ is singular.

**b) (2 points)** Explain how you can determine whether an upper triangular matrix $U$ is singular by only considering its diagonal entries. (Hint: If $U$ is singular, its columns are linearly dependent. Therefore $Ux = 0$ has a non-trivial solution.)