Midterm 2, Math 20F - Lecture B (Spring 2007)

1. An $n \times n$ matrix A is called skew-symmetric if $A^T = -A$. Specifically the set of 2×2 skew-symmetric matrices is given by

$$S^{2\times 2} = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \; : \; a \in \mathbb{R} \right\}.$$

Let $T \; : \; L^{2\times 2} \to S^{2\times 2}$ be the transformation from the 2×2 lower triangular matrices onto the 2×2 skew-symmetric matrices defined as

$$T\left(\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix},$$

where a, b, c are real numbers.

a) (1 point) Show that $S^{2\times 2}$ is a one-dimensional subspace of 2×2 matrices. Find a basis for this subspace.

Solution:
$S^{2\times 2}$ can be expressed as a span of a matrix,

$$S^{2\times 2} = \left\{ a \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; : \; a \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Therefore $S^{2\times 2}$ is a 1-dimensional subspace of the space of 2×2 matrices with the basis

$$\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

b) (2 points) Show that T is a *linear* transformation.

Solution:
We need to verify the following two properties to show that T is linear.

$$\begin{aligned} \text{Additivity}: \; T\left(\begin{bmatrix} b_1 & 0 \\ a_1 & c_1 \end{bmatrix} \right) + T\left(\begin{bmatrix} b_2 & 0 \\ a_2 & c_2 \end{bmatrix} \right) &= T\left(\begin{bmatrix} b_1 + b_2 & 0 \\ a_1 + a_2 & c_1 + c_2 \end{bmatrix} \right) \\ \begin{bmatrix} 0 & -a_1 \\ a_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a_2 \\ a_2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -(a_1 + a_2) \\ a_1 + a_2 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \text{Multiplication with a scalar}: \; T\left(\alpha \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) &= \alpha T\left(\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) \\ \begin{bmatrix} 0 & -\alpha a \\ \alpha a & 0 \end{bmatrix} &= \alpha \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \end{aligned}$$

c) (2 points) Find a basis for the kernel of T.

Solution:
By definition

$$\begin{aligned}
\text{Kernel(T)} &= \left\{ \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} : T\left(\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} : b, c \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : b, c \in \mathbb{R} \right\} \\
&= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.
\end{aligned}$$

Since the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not multiple of each other, they are linearly independent. Therefore the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for the kernel of T.

2. (Each part is 1 point) Determine whether each of the following statements is true or false. For each part circle either **T**(true) or **F**(false). You do not need to justify your answers.

(i) The matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is invertible. $\quad$ **T** $\quad$ **F**
True : The columns of the matrix are linearly independent.

(ii) Any $n \times n$ matrix A has an LU factorization, that is there exist a lower triangular matrix L and an upper triangular matrix U such that $A = LU$. $\quad$ **T** $\quad$ **F**
False : It may be necessary to reorder the rows of A so that an LU factorization exists.

(iii) The column space of an $n \times m$ matrix A is preserved when an elementary row operation is applied. That is if E is an $n \times n$ elementary matrix, the column spaces of EA and A are the same. $\quad$ **T** $\quad$ **F**
False : Row operations preserve the row space, but not the column space.

(iv) If the kernel of a linear transformation from a vector space to another vector space is $\{0\}$, then it is one-to-one. $\underline{\quad\textbf{T}\quad}$ $\underline{\quad\textbf{F}\quad}$
True : If $T(x) = T(y)$, by the linearity of T it follows that $T(x-y) = 0$. But the kernel of T is $\{0\}$ implying $x - y = 0$ or equivalently $x = y$; that is no two distinct x, y can be mapped to the same vector by T.

(v) The set $\{1, 1+x, 2+x^2, x+x^2\}$ is a basis for $\mathbb{P}_2 = \{a_0 + a_1x + a_2x^2 \ : \ a_0, a_1, a_2 \in \mathbb{R}\}$, the vector space of polynomials of degree at most two. $\underline{\quad\textbf{T}\quad}$ $\underline{\quad\textbf{F}\quad}$
False : $\mathbb{P}_2$ is a vector space of dimension 3. No basis can contain 4 vectors.

3. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

.

a) (2 points) Find an LU factorization for A.

Solution:
Recall the procedure to obtain an LU factorization. While we reduce A into an echelon form, we keep track of the elementary matrices applied to A from left, that is

$$[A|I] \longrightarrow [E_pE_{p-1}\dots E_1A|E_1^{-1}\dots E_{p-1}^{-1}E_p^{-1}] = [U|L]$$

where $U = E_pE_{p-1}\dots E_1A$ is the echelon matrix obtained by applying row operations corresponding to the elementary matrices $E_1, E_2, \dots E_p$ and $L = E_1^{-1}\dots E_{p-1}^{-1}E_p^{-1}$. Applying this procedure yields

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 := r_2 + 2r_1} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 := r_3 - 2r_2} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 2 & 1 \end{array}\right].$$

Therefore an LU factorization for A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

b) (3 points) Find the inverse of A.

Solution:
One approach is to use the LU factorization, since $A^{-1} = U^{-1}L^{-1}$ it would be sufficient to determine the inverses of U and L, then multiply them. Because of the special structure of the matrix (it has some zeros and the entries along any diagonal are fixed), it is also efficient to find the inverse of A directly by solving the linear systems

$$Ax_1 = e_1,\ Ax_2 = e_2,\ Ax_3 = e_3,$$

simultaneously. The matrix $[x_1\ x_2\ x_3]$ gives the inverse. Applying this approach yields

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 := r_2 + 2r_1} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 := r_3 - 2r_2} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 := -r_2\ r_3 := r_3/3} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & -4/3 & -2/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{r_2 := r_2 - r_3} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -4/3 & -2/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{r_1 := r_1 + r_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & -2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -4/3 & -2/3 & 1/3 \end{array}\right].$$

The matrix

$$A^{-1} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -2/3 & -1/3 & -1/3 \\ -4/3 & -2/3 & 1/3 \end{bmatrix}$$

is the inverse of A.

4. The singular matrices can also be characterized in terms of their LU factorizations.

a) (3 points) Suppose $A = LU$ is an LU factorization for an $n \times n$ matrix A. Given that L is invertible (the procedure based on the elementary matrices to find an LU factorization always yields an invertible lower triangular matrix) prove that A is singular if and only if U is singular.

Solution:

Suppose A is singular, then the columns of A are linearly dependent. Therefore there exists an $x \in \mathbb{R}^n$ not equal to zero such that

$$0 = Ax = LUx.$$

But L is invertible meaning

$$L^{-1}0 = L^{-1}LUx = Ux \Longrightarrow Ux = 0$$

for some non-zero x. This shows that the columns of U are linearly dependent or equivalently U is singular.

Now suppose U is singular. Since the columns of U must be linearly dependent

$$Ux = 0$$

for some vector $x \in \mathbb{R}^n$ not equal to zero. By multiplying both sides of the equation above by the matrix L from left we obtain

$$LUx = L0 = 0 \Longrightarrow Ax = 0$$

for some nonzero x implying that the columns of A are linearly dependent. Thus A is singular.

b) (2 points) Explain how you can determine whether an upper triangular matrix U is singular by only considering its diagonal entries. (Hint: If U is singular, its columns are linearly dependent. Therefore $Ux = 0$ has a non-trivial solution.)

Solution:

U is singular whenever at least one of its diagonal entries is equal to zero.

To see this let

$$U = [u_1 \; u_2 \; \ldots \; u_n],$$

that is $u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ are the columns of U. If the homogeneous system has a non-trivial solution, there exist scalars x_1, x_2, $\ldots$, x_n not all zero such that

$$x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = 0.$$

If x_n is non-zero, u_n can be written as a linear combination of $u_1, u_2, \ldots, u_{n-1}$. Since U is upper triangular, the last components of $u_1, u_2, \ldots, u_{n-1}$ are all equal to zero. But u_n can be written as a linear combination $u_1, u_2, \ldots, u_{n-1}$, so its last component, the diagonal entry u_{nn}, is also zero. If x_n is zero, let $j < n$ be the largest index such that $x_j \neq 0$. (that is $x_{j+1} = x_{j+2} = \cdots = x_n = 0$.) Since $x_1 u_1 + x_2 u_2 + \cdots + x_j u_j = 0$, from the previous argument, now the diagonal entry on the jth column, u_{jj}, is zero.

Examples:
The matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

are singular, while the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

is non-singular.