

Math 20F - Lecture C (Winter 2008)

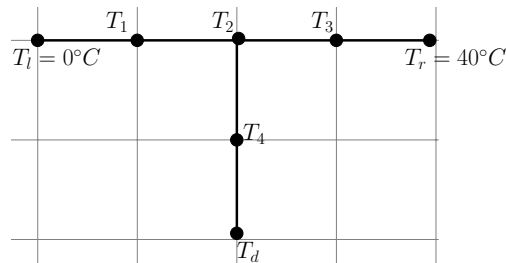
Homework 1

(Due on Friday, January 25th by 3pm)

**Please drop your homework off into the box on the 6th floor in the APM building. Note that there are separate boxes for the homeworks and Matlab assignments. Make sure to return it into the box for the homeworks and not for the Matlab assignments.*

**This homework covers sections 1.1-7 from your textbook. Some of the homework questions may require the knowledge of variety of topics and the ability to put them together. Ideally before attempting these questions, you should first practice with some of the suggested questions listed on the course homework page.*

1. (Sections 1.1, 1.2) In your physics lab you are given a T-shaped object illustrated below. You measure the temperatures at the left and right end-points as $T_l = 0^\circ C$ and $T_r = 40^\circ C$, respectively. You are asked to approximate the temperatures at the internal nodes by the average of the adjacent nodes. Note that there are three nodes adjacent to the 2nd node, so T_2 is the average of T_1 , T_3 and T_4 . All other nodes have two adjacent nodes.

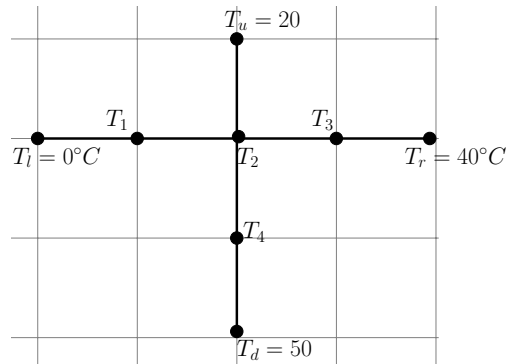


(a) Suppose you can also measure the temperature at the bottom as $T_d = 50^\circ C$. Set up a system of four linear equations in four unknowns. Solve it by row reducing the associated augmented matrix into the reduced echelon form to determine T_1, T_2, T_3, T_4 .

(b) Suppose your instrument breaks and you cannot measure the temperature at the bottom. Find all possible set of temperatures for the internal nodes. You can view T_d as a free variable; the temperatures at the internal nodes are basic variables that can be uniquely determined if T_d was given.

Key point : Part (b) asks you to set up an under-determined system with more unknowns than equations. Such a system cannot have a unique solution.

2. (Sections 1.1, 1.2) You are determined to make this process of estimating the temperatures more interesting. You put a vertical rod on top of a horizontal rod as illustrated in the figure. You measure the temperatures at the left and right end-points of the horizontal rod as $T_l = 0^\circ C$ and $T_r = 40^\circ C$ and the temperatures at the top and bottom end-points of the vertical rod as $T_u = 20^\circ C$ and $T_d = 50^\circ C$. You again aim to estimate the temperatures



at the internal nodes in the figure. Only now you assume that vertically the temperature at an internal node is given by the average of the adjacent nodes above and below the node. Similarly horizontally you assume that the temperature at an internal node is given by the average of the left and right adjacent nodes. Notice that there are two equations in which T_2 is involved. Vertically T_2 is the average of $T_u = 20^\circ C$ and T_4 . Horizontally T_2 is the average of T_1 and T_3 . At first glance your teaching assistant in the lab tells you that your assumptions are not realistic based on your measurements. Set up a linear system with the temperatures at the internal nodes as the unknowns and verify that the system is inconsistent.

Key point : The above problem asks you to set up an over-determined system with more equations than unknowns. Such a system is usually inconsistent, though it may rarely have a unique solution or infinitely many solutions.

3. (Section 1.1) In this question you will show that the row replacement operation preserves the solution set for a 2×2 system. Consider the linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

By multiplying the first equation by c and adding it to the second equation we obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ (a_{21} + ca_{11})x_1 + (a_{22} + ca_{12})x_2 &= (b_2 + cb_1). \end{aligned}$$

(a) Assume a point (x_1, x_2) satisfies the original linear system. Argue why it also satisfies the second system obtained by row-replacement.

(b) Assume a point (x_1, x_2) satisfies the system obtained by row-replacement. Show that it also satisfies the original system.

4. (Sections 1.2, 1.3, 1.7) Consider the 4×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

- (a) Find the pivot positions and pivot columns of A .
- (b) Do the columns of A span \mathbb{R}^4 ?
- (c) Are the columns of A linearly independent?

5. (Sections 1.3, 1.4, 1.5) Suppose the solution set for a homogeneous system $Ax = 0$ where A is a 3×3 matrix is given by

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Now given a vector $b \in \mathbb{R}^3$, a particular solution for $Ax = b$ is the vector

$$v_p = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

- (a) Find another solution of the system $Ax = b$ other than v_p .
- (b) Is the vector

$$v = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

a solution for the system $Ax = b$?

6. (Sections 1.1, 1.2, 1.3) In this question we want to determine whether there exists a quadratic polynomial in the form

$$p_2(x) = a_1 + a_2x + a_3x^2$$

that passes through the points $(1, 2)$ and $(-1, 4)$ and, if there are more than one such polynomials, characterize these polynomials.

- (a) Find a system of 2 linear equations in 3 unknowns whose solution set consists of all triples (a_1, a_2, a_3) such that $p_2(x) = a_1 + a_2x + a_3x^2$ passes through $(1, 2)$ and $(-1, 4)$. Provide also the 2×4 augmented matrix associated with this system.

(b) Row reduce the augmented matrix associated with the system of linear equations in part (a) into the reduced echelon form. Using the reduced echelon form decide whether there is a quadratic polynomial passing through $(1, 2)$ and $(-1, 4)$. If such a polynomial exists, provide one and determine whether it is the unique one. If there are more than one such polynomials, determine also the relation between the coefficients a_1, a_2 and a_3 so that the polynomial passes through $(1, 2)$ and $(-1, 4)$.

(c) Consider the quadratic polynomial with $a_1 = 0$ in the form

$$q_2(x) = a_2x + a_3x^2$$

and the vector

$$v_{a_2, a_3} = \begin{bmatrix} q_2(1) \\ q_2(-1) \end{bmatrix}.$$

If you can choose a_2 and a_3 arbitrary real numbers, identify the set of all possible vectors v_{a_2, a_3} by first expressing this set as a span of two vectors in \mathbb{R}^2 .

7. (Section 1.3) In this question you are expected to experiment with the Matlab routine provided generating the span of a pair of vectors $v1, v2$ both in \mathbb{R}^2 . Follow the instructions below to use this routine.

1. Open a Matlab command window.
2. Type (`>>` indicates the current command line; type only what follows `>>`)
`>> pwd`

Matlab will return you the directory in which you are working.

3. Copy the matlab routine *span.m* in this directory.
4. To plot the span of a pair of vectors $v1$ and $v2$, type
`>> span(v1, v2);`

Here you should make sure that both $v1$ and $v2$ are column vectors of size 2. For instance you can type

```
>> v1 = [2; 1];
>> v2 = [1; 2];
```

This will set

$$v1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad v2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Experiment with various $v1$ and $v2$. If the routine generates elliptic regions, the pair of vectors span all of \mathbb{R}^2 . (It generates elliptic regions, because the magnitudes of the vectors obtained as linear combinations are constrained to be less than 10.) Based on your experiments answer the following questions.

- (a) Determine a v_1 and v_2 such that $\text{span}(v_1, v_2) = 0$ that is the span of v_1 and v_2 consists of only the vector with both components equal to zero. Is there a unique v_1 and v_2 whose span is the zero vector or are there more than one such pair of vectors? Justify your answer.
- (b) Determine a pair of vectors v_1 and v_2 such that $\text{span}(v_1, v_2)$ is a line. In general when does the span of two vectors of size two form a line?
- (c) Determine a pair of vectors v_1 and v_2 such that $\text{span}(v_1, v_2) = \mathbb{R}^2$ that is all possible vectors of size 2. In general when is the span of two vectors of size two equal to \mathbb{R}^2 ?
- (d) Are there any other possibilities for the span of two vectors of size 2 other than the vector zero, a line and all of \mathbb{R}^2 ? Justify your answer.