Math 20F - Lecture C (Winter 2008)
Homework 2
(Due on Wednesday, February 13th by 3pm)

**Please drop your homework into the homework box on the 6th floor in the APM building.*

**This homework covers sections 2.1-3 and 2.5 from your textbook.*

**You don't need to turn in the questions with (*). But I highly recommend to solve these questions as well. These are proof questions. Working on them should give you insight about some most fundamental concepts that we couldn't touch on in class. Sometimes these exercises will let you make connections between seemingly unrelated topics.*

1. (Section 2.1) Solve questions 2.1.2, 2.1.6 and (*) 2.1.25 from your textbook.

2. (Section 2.1) Let

$$A = [a_1 \ a_2 \ \dots a_n] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{bmatrix}$$

and

$$R_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

be $n \times n$ matrices. Express the products AR_n and R_nA in terms of the columns $(a_1, a_2, \dots, a_n)$ of A and rows $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$ of A, respectively.

3. (Section 2.1) Let

$$A = [a_1 \ a_2 \ a_3 \ \dots \ a_n] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \vdots \\ \bar{a}_n \end{bmatrix}$$

be an $n \times n$ matrix where $a_1, a_2, \dots, a_n, \bar{a}_1^T, \bar{a}_2^T, \dots, \bar{a}_n^T \in \mathbb{R}^n$. In other words $a_1, a_2, \dots, a_n$ are the columns and $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ are the rows of A. Write down a matrix P such that

$$AP = [a_2 \ a_1 \ a_3 \dots \ a_n], \quad \text{and} \quad PA = \begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \\ \bar{a}_3 \\ \vdots \\ \bar{a}_n \end{bmatrix}.$$

Such a matrix is called a "permutation matrix", because it changes the order of the columns and rows of A when multiplied by A form the right and left, respectively.

4. (Section 2.2) Solve questions 2.2.3, 2.2.6, 2.2.21, 2.2.32, 2.2.35 from your textbook.

5. (Sections 2.2-3) An $n \times n$ matrix Q is called "orthogonal" if it satisfies the property $Q^T Q = I_n$. For instance an identity matrix of any size and the permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are orthogonal matrices.

(a) Give a 2×2 orthogonal matrix with nonzero entries (*i.e.* neither of the four entries is zero).

(b) Show that the columns of an orthogonal matrix are linearly independent.

(c) Given the linear system $Qx = b$ where Q is an $n \times n$ orthogonal matrix and $b \in \mathbb{R}^n$. Show that the solution x is unique and equal to

$$x = Q^T b \tag{1}$$

(d) For the 2×2 orthogonal matrix Q that you determined in **(a)** and

$$b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

solve the system $Qx = b$ by using equation (1). Verify also that your solution x satisfies $Qx = b$.

6. (Section 2.3) Solve questions 2.3.1, 2.3.4, 2.3.5 and (*) 2.3.18 from your textbook.

7. (Sections 2.3, 2.5) Suppose an LU factorization for the 3×3 matrix A is given as

$$A = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} -3 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{U}.$$

(a) In an LU factorization the lower triangular matrix is always invertible. Verify for A that the lower triangular matrix L is invertible.

(b) In general suppose for an $n \times n$ matrix $\tilde{A}$ an LU factorization is given by $\tilde{A} = \tilde{L}\tilde{U}$ where $\tilde{L}$ is invertible. Show that "$\tilde{U}$ is invertible if and only if $\tilde{A}$ is invertible".

(c) For A determine whether the upper triangular matrix U is invertible or not. Using the result of part **(b)** deduce whether A is invertible or not.

(d) Solve question 2.3.13 from your textbook, which asks you to identify which upper triangular matrices are invertible.

Remarks: As you show in the previous question LU factorization of a matrix reveals whether the matrix is invertible or not. But in practice LU factorization is most commonly employed to solve various linear systems with the same coefficient matrix, but different right-hand sides, that is a set of linear systems of the form $Ax = b_1$, $Ax = b_2$, $\ldots$, $Ax = b_m$. To determine whether a matrix is invertible in practice the "singular value decomposition" (see question 2.5.25 below) is a more efficient approach.

8. (Section 2.5) Solve questions 2.5.6, 2.5.9, 2.5.17, (*) 2.5.25 from your textbook.