Math 20F - Lecture C (Winter 2008)
Homework 3

(Due on Friday, February 29th by 3pm)

**Please return your homeworks to the homework box on the 6th floor in the APM building.*

**This homework covers sections 4.1-7 from your textbook.*

**You don't need to turn in the questions with (*). But I highly recommend to solve these questions as well.*

1. (Section 4.1) Solve questions 4.1.2, 4.1.8(*), 4.1.12(*) and 4.1.20 from the textbook.

2. (Section 4.2) Solve questions 4.2.4, 4.2.8(*), 4.2.18(*) and 4.2.32 from the textbook.

3. (Section 4.1, 4.3 and 4.5) Consider the subset of 3×3 matrices

$$\mathbb{B}^{3\times 3} = \left\{ \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}. \qquad (1)$$

Each matrix in $\mathbb{B}^{3\times 3}$ is called a *tridiagonal* matrix, since the entries along its main diagonal (*i.e.* $a_{ii} = a,\ i = 1, 2, 3$), subdiagonal (*i.e.* $a_{(i+1)i} = c,\ i = 1, 2$), superdiagonal (*i.e.* $a_{i(i+1)} = b,\ i = 1, 2$) are fixed.

(a) Explain why $\mathbb{B}^{3\times 3}$ is a subspace of 3×3 matrices.

(b) Find two different bases for $\mathbb{B}^{3\times 3}$. What is the dimension of the subspace of tridiagonal matrices?

4.(*) (Section 4.2) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{B}^{3\times 3}$,

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a & b-c & 0 \\ a & a & b-c \\ 0 & a & a \end{bmatrix}$$

where $\mathbb{B}^{3\times 3}$ denotes the set of 3×3 tridiagonal matrices defined in (1).

(a) Give a basis for the kernel of T.

(b) Give a basis for the range of T.

5. (Section 4.3) Solve questions 4.3.14, 4.3.20(*), 4.3.28(*), 4.3.32(*) and 4.3.34 from the textbook.

6. (Section 4.4) Solve questions 4.4.10, 4.4.12 from the textbook.

7. (Section 4.4-5 and 4.7) Please first read the additional material about the dimension of a vector space and change of basis. Choose any two bases say $B = \{b_1, b_2, \dots, b_n\}$ and $C = \{c_1, c_2, \dots, c_p\}$ for a vector space V. The proof of Theorem 1 shows that B and C must contain equal number of vectors (*i.e.* $n = p$) or otherwise one of B, C contains redundant vectors. If you are having difficult time with the proof, you can skip it. The essential point is that there is a one-to-one relation between the coordinates relative to the bases B and C. In particular

$$[v]_B = D[v]_C$$

where D is the change of coordinates matrix from C to B,

$$D = [\ [c_1]_B\ [c_2]_B\ \dots\ [c_n]_B\ ].$$

In class we saw that for the set of lower triangular matrices

$$\mathbb{L}^{2\times 2} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

two possible bases are

$$B_L = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$C_L = \left\{ \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

(a) Find the coordinates of the matrix

$$L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

relative to B_L.

(b) Find the change of coordinates matrix

$$D = \left[\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}_{B_L} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}_{B_L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{B_L} \right].$$

from C_L to B_L.

(c) Find the coordinates of the matrix L in **(a)** relative to C_L by exploiting the relation

$$[L]_{B_L} = D[L]_{C_L}.$$

You calculated the coordinate vector $[L]_{B_L}$ in **(a)** and the matrix D in **(b)**. Therefore to find $[L]_{C_L}$ you have to solve a 3×3 linear system.

8. (Section 4.5) Solve questions 4.5.4, 4.5.14 from the textbook.

9. (Section 4.6) Solve questions 4.6.4, 4.6.10, 4.6.14, 4.6.20(*) from the textbook.

10. (Section 4.6) The singular value decomposition of the matrix A is given by

$$A = \overbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\Sigma} \overbrace{\begin{bmatrix} 0.4\sqrt{2} & \frac{-1}{\sqrt{2}} & 0.3\sqrt{2} \\ 0.6 & 0 & -0.8 \\ 0.4\sqrt{2} & \frac{1}{\sqrt{2}} & 0.3\sqrt{2} \end{bmatrix}}^{V^T}.$$

where $V^TV = U^TU = I_3$. The orthogonality of U and V imply the linear independence of their columns. (You proved this in the second homework; see question **5.(b)** in hw 2.)

(a) Is A invertible? Explain. (Hint : Try to find a nonzero vector v such that $Av = 0$; pay attention to the rows of V^T.)

(b) Find a subset of the columns of V (or rows of V^T) that forms a basis for Null(A). (Hint : The set of columns of V is a basis for $\mathbb{R}^3$. You need to decide which columns of V are in the null space.)

(c) Find a subset of the columns of U that forms a basis for Col(A). (Hint : The set of columns of U is a basis for $\mathbb{R}^3$. You need to decide which columns of U are in the column space.)

(d) Based on your answers to **(b)** and **(c)** determine the rank of A.

Remark: The singular value decomposition is widely used to determine the rank of a matrix as well as to find bases for the null and column spaces. In particular Matlab to determine the rank of a matrix computes its singular value decomposition.