Dimension of a vector space: Two bases for a vector space V must consist of equal number of vectors as shown below. You can think each choice of basis as a selection of a coordinate system for V. The invariant property of V is the number of axes or the number of vectors in a basis, which we call the *dimension* of V.

Below the dimensions and bases are listed for the following vector spaces

Set of 2×2 matrices

$$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Set of 2×2 symmetric matrices

$$S^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Set of polynomials of degree at most n

$$\mathbb{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

Set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ - C.

Vector Space	A Basis	Dimension
$\mathbb{R}^{2\times 2}$	$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	4
$S^{2\times 2}$	$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	3
$\mathbb{P}_n$	$\{1, x, x^2, \ldots, x^n\}$	n+1
C	No finite basis	infinite dimensional

Theorem 1. *Two bases for V consist of equal number of vectors.*

Proof. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V. In class to justify the invariance of the number of elements in a basis for V, we introduced the coordinate mapping $T : V \to \mathbb{R}^n$

$$T(v) = [v]_B$$

where $[v]_B$ is the coordinate vector of v relative to B. The coordinate mapping T is linear and one-to-one. Also the kernel of T is $\{0\}$.

Suppose $C = \{c_1, c_2, \ldots, c_n, c_{n+1}, \ldots, c_{n+k}\}$ is another basis for V. We need to show that the vectors $c_{n+1}, \ldots, c_{n+k}$ are redundant, *i.e.* the reduced set $C^r = \{c_1, c_2, \ldots, c_n\}$ spans V. Since C is a basis, the set C^r is linearly independent. It turns out that the image of the set

$$T(C^r) = \{T(c_1), T(c_2), \ldots, T(c_n)\}$$

is also linearly independent (Try to show this as an optional exercise). Indeed n linearly independent vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ meaning $T(C^r)$ is a basis for $\mathbb{R}^n$.

Any vector $v \in V$ is in the span of B, that is

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n, \quad \text{and} \quad [v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

But since $T(C^r)$ is a basis for $\mathbb{R}^n$, for some scalars $\beta_1, \beta_2, \ldots, \beta_n$

$$\begin{aligned} T(v) = [v]_B &= \beta_1 T(c_1) + \beta_2 T(c_2) + \cdots + \beta_n T(c_n) \\ &= T(\beta_1 c_2 + \beta_2 c_2 + \cdots + \beta_n c_n) \end{aligned}$$

holds. But T is one-to-one (*i.e.* $T(v) = T(w) \ \Rightarrow \ v = w$), so

$$v = \beta_1 c_2 + \beta_2 c_2 + \cdots + \beta_n c_n \quad \Rightarrow [v]_{C^r} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Let $D = [T(c_1)\ T(c_2)\ \ldots\ T(c_n)]$. Notice that D is an $n \times n$ matrix and invertible, because its columns are linearly independent. We deduce that

$$[v]_B = \beta_1 T(c_1) + \beta_2 T(c_2) + \cdots + \beta_n T(c_n) = D[v]_{C^r}.$$

This establishes a one-to-one relation between the coordinates relative to B and C^r.

Choose any $v \in V$ with the coordinate vector $[v]_B$. But then the coordinates relative to C is given by

$$[v]_C = D^{-1}[v]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

We conclude that

$$v = \beta_1 c_1 + \beta_2 c_2 + \ldots \beta_n c_n.$$

Therefore $v \in \text{span}(C^r)$. Since the above argument applies to any $v \in V$, the set C^r spans V and therefore is a basis. The elements $c_{n+1}, \ldots, c_{n+k}$ of C are redundant, indeed depend on the first n vectors in C.

Above we assumed that C contains more vectors than B. If we assume C contains fewer vectors than B, then B contains more vectors. The above argument with the roles of B and C interchanged can be used to show that the extra vectors in B are redundant. □

Remark: (Change of Basis briefly - section 4.7 in the textbook)
Let $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ be bases for V and T be the coordinate mapping relative to B, *i.e.* $v \to [v]_B$. The proof above is based on the construction of the matrix

$$D = [T(c_1)\ T(c_2)\ \ldots\ T(c_n)] = [\ [c_1]_B\ [c_2]_B\ \ldots\ [c_n]_B\ ].$$

This matrix is called the change of coordinates matrix from C to B as

$$[v]_B = D[v]_C,$$

which can be used to find the coordinate vector of v relative to B given the coordinates relative to C. Clearly the change of coordinates matrix from B to C is D^{-1} and

$$[v]_C = (D^{-1})[v]_B.$$