Additional Questions - Math 20F, Linear Algebra

1. We have seen the following factorizations.

$$\begin{aligned} \text{LU factorization} \quad &: \quad A = LU \\ \text{Eigenvalue decomposition} \quad &: \quad A = VDV^{-1} \\ \text{QR factorization} \quad &: \quad A = QR \\ \text{Singular value decomposition} \quad &: \quad A = \hat{U}\Sigma\hat{V}^T \end{aligned}$$

where L is a lower triangular matrix, U, R are upper triangular matrices, $Q, \hat{U}, \hat{V}$ are orthogonal matrices, V is invertible and has the eigenvectors of A as its columns and D, Σ are diagonal matrices.

(a) Suppose you want to solve various linear systems with the same coefficient matrix

$$Ax = b_1, \; Ax = b_2, \; \ldots, \; Ax = b_n.$$

If you were given the factorizations above, which of them would help to *solve the linear systems* above more efficiently.
b) Which of the factorizations is convenient to calculate the *matrix powers*?
c) Which of the factorizations do provide an *orthonormal basis for the column space* of A?
d) Which of the factorizations would you use to find the *determinant* of A?
e) Which of the factorizations do reveal the *rank* of the matrix?

2. Let $c_1, c_2, \ldots, c_{n-1}$ be real numbers. How are the eigenvalues and eigenvectors of A and $c_1A + c_2A^2 + \cdots + c_{n-1}A^{n-1}$ related?

3. Find the eigenvalues, eigenvectors and eigenspaces of

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Is J diagonalizable?

4. Let $u_1, u_2, v_1, v_2 \in \mathbb{R}^3$. Assume the sets $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are linearly independent.

$$A = u_1v_1^T + u_2v_2^T$$

a) Determine the rank of A.
b) Determine the dimension of the null space of A.
c) Determine a basis for the column space of A.

5. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

a) Evaluate the determinant of A using row-reduction.
b) Evaluate the determinant of B using a cofactor expansion.
c) Based on the determinants in part **a)** and **b)** are A and B invertible?
d) Evaluate the determinant of AB.
e) Evaluate the determinant of $A+B$.
f) Find a basis for the null-space of A.
g) Find a basis for the column-space of A.
h) Find a basis for the row-space of A.

6. The set

$$\mathbb{P} = \text{span}\{1 + x^2, 2x^2, 2\}$$

is a subspace of the polynomials of degree at most two, $\mathbb{P}_2 = \{c_1 + c_2 x + c_3 x^2 \ : \ c_1, c_2, c_3 \in \mathbb{R}\}$. What is the dimension of $\mathbb{P}$? Find a basis for this subspace.

7. Consider the transformation from the set of polynomials of degree at most two to the set of polynomials of degree at most three

$$T : \mathbb{P}_2 \longrightarrow \mathbb{P}_3, \quad T(c_1 + c_2 x^2) = c_2 + c_1 + (c_2 + c_1)x^3$$

a) Show that T is linear.
b) Find a basis for the range of T.
c) Find a basis for the kernel of T.

8. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

a) Show that the overdetermined system $Ax = b$ has no solution.
b) Explain why a least squares solution $\hat{x}$ satisfying

$$\|b - A\hat{x}\| \leq \|b - Ax\|, \quad \text{for all } x$$

is not unique.
c) Find a basis for the set of solutions for the least squares problem above.

9. Which of the following sets are subspaces? Justify your answers.

a) The subset of polynomials of degree at most 2,

$$\{2 + c_1 x + c_2 x^2 : c_1, c_2 \in \mathbb{R}\}$$

b) Let W be a subspace of $\mathbb{R}^n$. The set consisting of vectors that are perpendicular to all of the vectors in W. (This set is called the *orthogonal complement* of W.)
c) The subset of 2×2 matrices,

$$\left\{ \begin{bmatrix} c_1 & c_1 c_2 \\ c_1 c_2 & c_2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

d) The subset of $\mathbb{R}^3$,

$$\left\{ \begin{bmatrix} c_1 \\ c_1 + c_2 \\ c_2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

10. Find an eigenvalue decomposition for

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

Using the eigenvalue decomposition calculate A^5.

11. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

a) Is the set $\{v_1, v_2, v_3\}$ linearly independent?
b) Is the set $\{v_1, v_2, v_3\}$ orthogonal?
c) Find an orthogonal basis for the subspace $\text{span}\{v_1, v_2, v_3\}$.

12. **(*)** In this question if you can complete the following steps you will show that the reduced echelon form for a matrix A is unique. Assume R_1 and R_2 are reduced echelon forms obtained by applying row operations, that is

$$E_p E_{p-1} \dots E_1 A = R_1$$
$$\tilde{E}_k \tilde{E}_{k-1} \dots \tilde{E}_1 A = R_2,$$

where $E_1, E_2, \dots, E_p, \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_k$ are elementary matrices and therefore invertible. We need to show that $R_1 = R_2$.

a) Show that applying a row operation preserves the rank that is

$$\mathrm{rank}(EA) = \mathrm{rank}(A)$$

where E is an elementary matrix. This would show that A, R_1 and R_2 have the same rank.
b) Explain why R_1 and R_2 must have equal number of pivot columns. (Hint: Since R_1 and R_2 are of same rank, the dimensions of their null spaces are also equal.)
c) By showing that R_1 and R_2 have the same pivot positions, conclude that R_1 and R_2 have exactly the same pivot columns. You need to verify that the (i, j)th entry of R_1 is a pivot position if and only if the (i, j)th entry of R_2 is a pivot position. (Hint: Since R_1 and R_2 are of same rank, R_1 and R_2 have equal number of zero rows below the non-zero rows. If the pivot positions in the first rows of R_1 and R_2 were different, the row spaces of R_1 and R_2 would be different. Now show that the pivot positions on the second rows should be the same given that the pivot positions on the first rows are the same. Finally generalize this; assume that the first k rows of R_1 and R_2 have exactly the same pivot positions, show that $k + 1$th rows must have the same pivot positions.)
d) Using the fact that a non-pivot column can be written as a linear combination of the previous columns deduce that the non-pivot columns of R_1 and R_2 must also be the same.