Name:	**PID:**

Discussion Section		
No:	**Time:**	**TA:**

Midterm 1, Math 20F - Winter 2008

Duration: 100 minutes
This is an open book exam. Calculators are not allowed.
To get full credit you should support your answers.

1. Consider the system of linear equations

$$\begin{aligned} x_1 - 3x_2 &= 4 \\ 3x_1 - 6x_2 &= 3 \\ 2x_1 - hx_2 &= b \end{aligned}$$

where h and b are scalars.

(a) (1 point) Explain why the system cannot have infinitely many solutions.

Solution:
Geometrically the first two equations can only be satisfied by a unique point, since they represent lines not parallel to each other. Third equation (depending on the choice of h and b) can be a line that either goes through this intersection point in which case the system has a "unique solution". If the line represented by the third equation does not pass through the intersection of the first two, the system is "inconsistent". There are only two possibilities.

This can also be seen by row-reducing the augmented matrix into an echelon form and verifying that the system has no free variables.

$$\begin{bmatrix} 1 & -3 & 4 \\ 3 & -6 & 3 \\ 2 & -h & b \end{bmatrix} \xrightarrow[r_2 := r_2 - 3r_1,\ r_3 := r_3 - 2r_1\ r_2 := r_2/3]{} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -3 \\ 0 & -h+6 & b-8 \end{bmatrix}$$

If $h = 6$, then the entry $(2, 2)$ is a pivot entry. Otherwise the entry $(3, 2)$ is non-zero, but can be made zero by applying one more row-replacement operation. Therefore the first two columns (the columns corresponding to variables) are pivot columns. There are no free variables; the solution may be unique or may not exist at all.

Answer for the other type: The only difference in the question is that the third equation is $-4x_1 + hx_2 = b$. Same reasoning applies both geometrically and algebraically. Row-reduction again reveals that there cannot be free variables.

(b) (2 points) For what values of h and b is the system inconsistent?

Solution The system is inconsistent whenever the last column of the augmented matrix is a pivot column. From the row-reduced matrix in **(a)** if $h = 6$, then the last column is a pivot column whenever $b \neq 8$. If $h \neq 6$, then applying one more row-replacement operation yields

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -3 \\ 0 & -h+6 & b-8 \end{bmatrix} \xrightarrow[r_3 := r_3 + (h-6)r_2]{} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & b-8-3(h-6) \end{bmatrix}.$$

which implies that the system is inconsistent, when $b \neq 8 + 3(h-6)$. To summarize the system is inconsistent, whenever (regardless of $h = 6$ or not)

$$b \neq 8 + 3(h-6)$$

Answer for the other type: $b \neq -16 - 3(h-12)$.

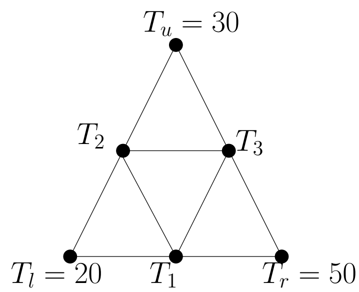
(c) (1 point) For what values of h and b does the system have a unique solution

Solution: This is the opposite of **(a)**. We want the last column to be a non-pivot column, which is the case when

$$b = 8 + 3(h-6).$$

Answer for the other type: $b = -16 - 3(h-12)$.

2. In this question for the triangular shape below we want to estimate the temperatures T_1, T_2 and T_3 at the internal nodes. The temperatures at the corners of the triangle are given as 20, 30 and 50 as shown in the figure below. The temperature at each internal node is estimated by the average of the temperatures at the four adjacent nodes. Two nodes are said to be adjacent if they are connected by an edge.



(a) (2 points) Set up a linear system of three equations with unknowns T_1, T_2 and T_3. Write down also the augmented matrix.

Solution: Equations are as follows

- First node : $(T_2 + T_3 + 20 + 50)/4 = T_1 \Rightarrow 4T_1 - T_2 - T_3 = 70$
- Second node : $(T_1 + T_3 + 20 + 30)/4 = T_2 \Rightarrow 4T_2 - T_1 - T_3 = 50$
- Third node : $(T_2 + T_1 + 30 + 50)/4 = T_3 \Rightarrow 4T_3 - T_1 - T_2 = 80$

Augmented matrix for the system :

$$\begin{bmatrix} 4 & -1 & -1 & 70 \\ -1 & 4 & -1 & 50 \\ -1 & -1 & 4 & 80 \end{bmatrix}$$

Answer for the other type: System of linear equations :

$$\begin{aligned} 4T_1 - T_2 - T_3 &= 50 \\ 4T_2 - T_1 - T_3 &= 30 \\ 4T_3 - T_1 - T_2 &= 60 \end{aligned}$$

Augmented matrix :

$$\begin{bmatrix} 4 & -1 & -1 & 50 \\ -1 & 4 & -1 & 30 \\ -1 & -1 & 4 & 60 \end{bmatrix}$$

(b) (2 points) By reducing the augmented matrix into the reduced echelon form solve the linear system from part **(a)**.

Solution: Row-reduce the augmented matrix into the reduced echelon form

$$\begin{bmatrix} 4 & -1 & -1 & 70 \\ -1 & 4 & -1 & 50 \\ -1 & -1 & 4 & 80 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} -1 & 4 & -1 & 50 \\ 4 & -1 & -1 & 70 \\ -1 & -1 & 4 & 80 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & -1 & 50 \\ 4 & -1 & -1 & 70 \\ -1 & -1 & 4 & 80 \end{bmatrix} \xrightarrow{r_2 := r_2 + 4r_1,\ r_3 := r_3 - r_1} \begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & 15 & -5 & 270 \\ 0 & -5 & 5 & 30 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & 15 & -5 & 270 \\ 0 & -5 & 5 & 30 \end{bmatrix} \xrightarrow{r_2 := r_2/5,\ r_3 := r_3/5,\ r_2 \leftrightarrow r_3} \begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & -1 & 1 & 6 \\ 0 & 3 & -1 & 54 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & -1 & 1 & 6 \\ 0 & 3 & -1 & 54 \end{bmatrix} \xrightarrow{r_3 := r_3 + 3r_2,\ r_3 := r_3/2} \begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & 1 & 36 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & -1 & 50 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & 1 & 36 \end{bmatrix} \xrightarrow{r_2 := r_2 - r_3,\ r_1 := r_1 + r_3} \begin{bmatrix} -1 & 4 & 0 & 86 \\ 0 & -1 & 0 & -30 \\ 0 & 0 & 1 & 36 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 & 0 & 86 \\ 0 & -1 & 0 & -30 \\ 0 & 0 & 1 & 36 \end{bmatrix} \xrightarrow{r_1 := r_1 + 4r_2,\ r_2 := -r_2,\ r_1 := r_1} \begin{bmatrix} 1 & 0 & 0 & 34 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 36 \end{bmatrix}$$

The temperatures at the internal nodes are $T_1 = 34$, $T_2 = 30$ and $T_3 = 36$.

Answer for the other type: $T_1 = 24$, $T_2 = 20$ and $T_3 = 26$.

3. Let

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad a_3 = \begin{bmatrix} 1 \\ -1 \\ -8 \end{bmatrix}.$$

(a) (1 point) Give geometric descriptions of the sets span$\{a_1\}$ and span$\{a_1, a_2\}$.

Solution: Span of a non-zero vector, in particular a_1, is a line through the origin. Span of two linearly independent vectors, in particular a_1 and a_2, is a plane through the origin. The vectors a_1, a_2 are linearly independent, since they are not multiples of each other

Answer for the other type: span$\{a_1\}$ is a line through the origin. span$\{a_1, a_2\}$ is a plane through the origin.

(b) (2 points) Is the set $\{a_1, a_2, a_3\}$ linearly independent?

Solution: The set is linearly independent if and only if the system

$$[a_1\ a_2\ a_3]x = 0$$

has no free variables. Row-reducing the coefficient matrix into the echelon form we see that there are free variables.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & -1 \\ 2 & -1 & -8 \end{bmatrix} \xrightarrow[r_2 := r_2 - 3r_1,\ r_3 := r_3 - 2r_1]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -4 \\ 0 & -5 & -10 \end{bmatrix} \xrightarrow[r_3 := r_3 - 5/2r_2]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Third column is a non-pivot column. The system has a free variable implying the set $\{a_1, a_2, a_3\}$ is linearly dependent.

Answer for the other type: The vectors

$$a_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad a_3 = \begin{bmatrix} -4 \\ -6 \\ 7 \end{bmatrix}$$

are linearly dependent. Indeed $2a_1 - 3a_2 = a_3$.

(c) (1 point) Does the set $\{a_1, a_2, a_3\}$ span $\mathbb{R}^3$? Give a geometric description of span$\{a_1, a_2, a_3\}$?

Solution: A set of n vectors span $\mathbb{R}^n$ if and only if the set is linearly independent. Since $\{a_1, a_2, a_3\}$ is linearly dependent, its span is not $\mathbb{R}^3$. Indeed a_1, a_2, a_3 are vectors lying on the same plane through the origin in $\mathbb{R}^3$. Their span generates the plane on which they lie.

Answer for the other type: The set $\{a_1, a_2, a_3\}$ cannot span $\mathbb{R}^3$. Since they are linearly dependent, their span is a plane through the origin.

4. The solution set of a linear system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b$$

of "three equations in four unknowns" is given as

$$\begin{aligned} x_1 &= x_2 + 2x_3 + 2 \\ x_2,\ x_3 &\text{ are free} \\ x_4 &= 1. \end{aligned}$$

(a) (1 point) Give a particular solution for the linear system.

Solution: We can assign any values to free variables. For instance $x_2 = x_3 = 0$ yields the particular solution $x_1 = 2$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$ or in vector notation

$$v_p = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Answer for the other type: For instance $x_1 = 4$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 3$ or in vector notation

$$v_p = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

is a particular solution.

(b) (1 point) Determine the pivot columns of the 3×4 coefficient matrix A for the linear system.

Solution: The second, third columns are non-pivot columns, because they correspond the free variables x_2 and x_3. The first and fourth columns are pivot columns.

Answer for the other type: The first and fourth columns are pivot columns.

(c) (2 points) Write down the solution set for the associated homogeneous system as a span of set of vectors.

Solution: Both of the equations

$$A\begin{bmatrix}2\\0\\0\\1\end{bmatrix} = b \text{ and } A\begin{bmatrix}x_2+2x_3+2\\x_2\\x_3\\1\end{bmatrix} = b$$

hold because the vectors with which we multiply A are solutions to the linear system. Subtracting the first equation from the second we obtain

$$A\begin{bmatrix}x_2+2x_3\\x_2\\x_3\\0\end{bmatrix} = 0.$$

Therefore the solution set for the homogeneous system is

$$\left\{\begin{bmatrix}x_2+2x_3\\x_2\\x_3\\0\end{bmatrix} : x_2, x_3 \in \mathbb{R}\right\} = \left\{x_2\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}2\\0\\1\\0\end{bmatrix} : x_2, x_3 \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\end{bmatrix}\right\}$$

Answer for the other type: Solution set for the homegeneous system:

$$\operatorname{span}\left\{\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\right\}$$

5. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}.$$

(a) (2 points) Find the transformation matrix for T, that is find a matrix A such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}.$$

Solution:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix} = x_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transformation matrix :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Answer for the other type: The transformation matrix :

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

(b) (1 point) Is the transformation T one-to-one? Justify your answer.

Solution: The columns of the transformation matrix A of part **(a)** are not multiples of each other; they are linearly independent. Therefore T is one-to-one.

Answer for the other type: The transformation is one-to-one, since the columns of the transformation matrix are linearly independent.

(c) (1 point) Is the transformation T onto $\mathbb{R}^3$? Justify your answer.

Solution: The transformation matrix A has two columns. Two vectors cannot span $\mathbb{R}^3$. Since the columns of A do not span $\mathbb{R}^3$, T is not onto $\mathbb{R}^3$.

Answer for the other type: The transformation is not onto $\mathbb{R}^3$, because the columns of the transformation matrix do not span $\mathbb{R}^3$.