Solutions to Homework 4

1.

(a) $A_1 = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible,

since $\det(A_1) = (3)(1)(2) = 6 \neq 0$

$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ is singular,

because $\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{a_1} + \underbrace{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}_{a_2} - \underbrace{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}}_{a_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

that is the cols of A_2 are lin. dependent.

$A_3 = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ is singular,

since $\det(A_3) = (3)(0)(2) = 0$

$A_4 = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ is invertible,

since $\det(A_4) = (3)(1)(2) = 6 \neq 0$

(b) The solution to $A_1 x = b$ is unique, since A_1 is invertible

$$\underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}}_{b}$$

Back substitution

Eq (3): $2x_3 = 2 \Longrightarrow x_3 = 1$

Eq (2): $x_2 - x_3 = -3 \Longrightarrow x_2 - 1 = -3 \Longrightarrow x_2 = -2$

Eq (1): $3x_1 - x_2 + 2x_3 = -6 \Longrightarrow 3x_1 + 2 + 2 = -6 \Longrightarrow x_1 = -10/3$

The solution to $A_4 x = b$ is also unique, i.e A_4 is also invertible

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}}_{A_4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}}_{b}$$

Forward Substitution

(1): $3x_1 = -6 \Longrightarrow x_1 = -2$

(2): $2x_1 + x_2 = -3 \Longrightarrow -4 + x_2 = -3 \Longrightarrow x_2 = 1$

(3): $x_1 + 2x_2 + 2x_3 = 2 \Longrightarrow -2 + 2 + 2x_3 = 2 \Longrightarrow x_3 = 1$

Since A_2 and A_3 are singular, the solutions to $A_2 x = b$ and $A_3 x = b$ are not unique.

2. Suppose $Ax=b$ has a unique soln.
Assume $y \in \mathbb{R}^n$ is s.t. $Ay=0$. Then

$$Ax + Ay = b + 0$$
$$\Longrightarrow A(x+y) = b.$$

But the linear system has a unique soln implying

$$x = x+y \Longrightarrow y = 0$$

This shows that $Ay=0$ only for $y=0$; equivalently the cols of A are lin. independent.
Therefore A is invertible.

3.

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Augmented matrix and row reduction

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 2 & 1 & -1 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \xrightarrow[r_2 := r_2 - 2r_1]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 3 & -1 & -7 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 := r_3 - \frac{2}{3} r_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 3 & -1 & -7 \\ 0 & 0 & 5/3 & 15/3 \end{array}\right]$$

By back substitution

$$\tfrac{5}{3} x_3 = \tfrac{15}{3} \implies x_3 = 3$$

$$3x_2 - x_3 = -7 \implies x_2 = -4/3$$

$$x_1 - x_2 = 3 \implies x_1 = 5/3$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 8 & 8 & 6 \\ 4 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 14 \\ 3 \end{bmatrix}$$

Augmented matrix and row reduction

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 8 & 8 & 6 & 14 \\ 4 & 6 & 1 & 3 \end{array}\right] \xrightarrow[\substack{r_2 := r_2 - 4r_1 \\ r_3 := r_3 - 2r_1}]{} \left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 0 & -12 & -6 & 6 \\ 0 & -4 & -5 & -1 \end{array}\right]$$

$$\xrightarrow[r_3 := r_3 - \frac{1}{3} r_2]{} \left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 0 & -12 & -6 & 6 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

By back substitution

$$-3x_3 = -3 \implies x_3 = 1$$

$$-12x_2 - 6x_3 = 6 \implies x_2 = -1$$

$$2x_1 + 5x_2 + 3x_3 = 2 \implies x_1 = 2$$

4. Exercise 1.2.19

(a) By the Kirchoff's law the total voltage around each mesh must be zero. Therefore we obtain the following system of 9 linear equations in 9 unknowns.

Upper left-most mesh

$$1(x_1-x_4)+1(x_1-x_2)+1x_1=0$$

Upper mesh in the middle

$$1(x_2-x_1)+1(x_2-x_5)+1(x_2-x_3)+1x_2=0$$

Upper right-most mesh

$$1(x_3-x_2)+1(x_3-x_6)+1x_3=0$$

Left-most mesh in the middle row

$$1x_4+1(x_4-x_7)+1(x_4-x_5)+1(x_4-x_1)=0$$

Middle mesh in the middle row

$$1(x_5-x_4)+1(x_5-x_8)+1(x_5-x_6)+1(x_5-x_2)=0$$

Right-mosh mesh in the middle row

$$1(x_6-x_5)+1(x_6-x_9)+1x_6+1(x_6-x_3)=0$$

Bottom left-most mesh

$$1x_7+1(x_7-x_8)+1(x_7-x_4)=0$$

Bottom mesh in the middle

$$1(x_8-x_7)+1x_8+1(x_8-x_9)+1(x_8-x_5)=0$$

Bottom right-most mesh

$$1(x_9-x_8)+1x_9-9+1(x_9-x_6)=0$$

Resulting linear system

$$\begin{bmatrix} 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

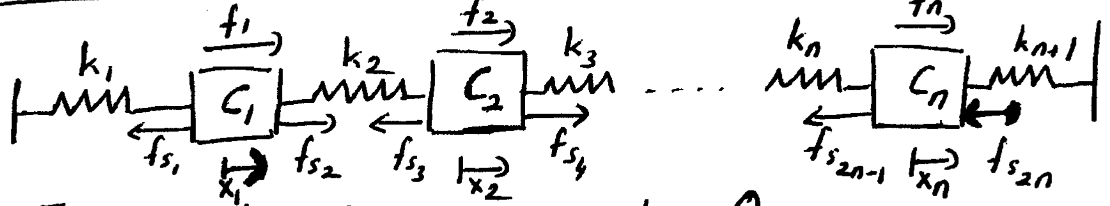
(b) The solution is

$$x = [\,0.3 \quad 0.45 \quad 0.6 \quad 0.45 \quad 0.9 \quad 1.35 \quad 0.6 \quad 1.35 \quad 3.9\,]^T$$

See the attached Matlab output.

4. Exercise 1.2.20

(a)



Forces on C_1 sum up to 0

$$f_1 - f_{s_1} + f_{s_2} = f_1 - k_1 x_1 + k_2 (x_2 - x_1) = 0$$

Forces on C_2

$$f_2 - f_{s_3} + f_{s_4} = f_2 - k_2(x_2 - x_1) + k_3 (x_3 - x_2) = 0$$

Forces on C_i where $1 < i < n$

$$f_i - f_{s_{2i-1}} + f_{s_{2i}} = f_i - k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) = 0$$

Forces on C_n

$$f_n - f_{s_{2n-1}} + f_{s_{2n}} = f_n - k_n(x_n - x_{n-1}) - k_{n+1}x_n = 0$$

Resulting linear system

$$\overbrace{\begin{bmatrix} k_1+k_2 & -k_2 & & & & 0 \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & -k_3 & k_3+k_4 & -k_4 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & -k_{n-1} & k_{n-1}+k_n & k_n \\ & & & & -k_n & k_n+k_{n+1} \end{bmatrix}}^{A} \times \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \overbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}^{F}$$

Note: for the coefficent matrix above

$a_{ij} = 0$ for all i,j s.t. $|i-j| \geqslant 2$

(b) With the given values of k_i and f_i the coeffricent matrix is in the form

$$A = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ f_5 \\ \\ \\ \\ f_{16} \\ \\ \\ \\ \end{matrix}$$

The solution rounded to two decimal digits

$$x = [0.52\ 1.05\ 1.57\ 2.10\ 2.62\ 2.14\ 1.67\ 1.19\ 0.71\ 0.24\ -0.24\ -0.71\ -1.19\ -1.67\ -2.14\ -2.62\ -2.10\ -1.57\ -1.05\ -0.52]^T$$

See the attached Matlab output.

5. (a) Given $A \in \mathbb{R}^{n \times n}$ (an upper tri. matrix), $b \in \mathbb{R}^n$

for $i = n, n-1, \dots, 1$

  $x_i \leftarrow b_i$

  for $j = i+1, i+2, \dots, n$ (not executed for $i=n$)

    $x_i \leftarrow x_i - a_{ij} \cdot x_j$

  end

  $x_i \leftarrow x_i / a_{ii}$

end

$$\boxed{\begin{array}{l}\text{General formula for back substitution for } i=1,\dots,n \\ x_i = \left(b_i - a_{i,i+1}x_{i+1} - a_{i,i+2}x_{i+2} - \cdots - a_{i,n}x_n\right)/a_{ii}\end{array}}$$

(b) For each $i = 1, 2, \dots, n$ the inner loop requires $2(n-i)$ flops.

For each i also a division is performed

$$\begin{aligned}\text{Total \# of Flops} &= \sum_{i=1}^{n} (2(n-i)+1) \\ &= 2\sum_{i=1}^{n} n + \sum_{i=1}^{n} 1 - 2\sum_{i=1}^{n} i \\ &= 2n^2 + n - \cancel{2}\frac{n(n+1)}{\cancel{2}} = n^2\end{aligned}$$

(c) See the attached print-out

(d) See the attached Matlab output.

Matlab Output Q4(Exercise 1.2.19) - part (b)
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```
>> A = [3 -1 0 -1 0 0 0 0 0;
-1 4 -1 0 -1 0 0 0 0;
0 -1 3 0 0 -1 0 0 0;
-1 0 0 4 -1 0 -1 0 0;
0 -1 0 -1 4 -1 0 -1 0;
0 0 -1 0 -1 4 0 0 -1;
0 0 0 -1 0 0 3 -1 0;
0 0 0 0 -1 0 -1 4 -1;
0 0 0 0 0 -1 0 -1 3]

A =

     3    -1     0    -1     0     0     0     0     0
    -1     4    -1     0    -1     0     0     0     0
     0    -1     3     0     0    -1     0     0     0
    -1     0     0     4    -1     0    -1     0     0
     0    -1     0    -1     4    -1     0    -1     0
     0     0    -1     0    -1     4     0     0    -1
     0     0     0    -1     0     0     3    -1     0
     0     0     0     0    -1     0    -1     4    -1
     0     0     0     0     0    -1     0    -1     3

>> b = [0; 0; 0; 0; 0; 0; 0; 0; 9]

b =

     0
     0
     0
     0
     0
     0
     0
     0
     9

>> x = A\b

x =

    0.3000
    0.4500
    0.6000
    0.4500
    0.9000
    1.3500
```

Matlab Output Q4 (Exercise 1.2.19) - part (b)
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```
    0.6000
    1.3500
    3.9000

>> A*x

ans =

    0.0000
   -0.0000
   -0.0000
         0
    0.0000
         0
    0.0000
    0.0000
    9.0000
```

Matlab Output Q4 (Exercise 1.2.20) - part (b)
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```
     0     0     0     0
     0     0     0     0
     0     0     0     0
     0     0     0     0
    -1     0     0     0
     2    -1     0     0
    -1     2    -1     0
     0    -1     2    -1
     0     0    -1     2

>> f = zeros(20,1);
>> f(5) = 1;
>> f(16) = -1;
>> x = A\f

x =

     5.238095238095236e-01
     1.047619047619047e+00
     1.571428571428571e+00
     2.095238095238094e+00
     2.619047619047618e+00
     2.142857142857141e+00
     1.666666666666666e+00
     1.190476190476190e+00
     7.142857142857136e-01
     2.380952380952371e-01
    -2.380952380952394e-01
    -7.142857142857159e-01
    -1.190476190476192e+00
    -1.666666666666669e+00
    -2.142857142857146e+00
    -2.619047619047622e+00
    -2.095238095238097e+00
    -1.571428571428573e+00
    -1.047619047619049e+00
    -5.238095238095243e-01

>> A*x

ans =

    -2.220446049250313e-16
     4.440892098500626e-16
                         0
    -4.440892098500626e-16
```

Matlab Output Q4 (Exercise 1.2.20) - part (b)
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```
>> C = zeros(20,1);
>> C(1) = 2;
>> C(2) = -1;
>> R = zeros(1,20);
>> R(1,1) = 2;
>> R(1,2) = -1;
>> A = toeplitz(C,R)

A =

  Columns 1 through 16

   2  -1   0   0   0   0   0   0   0   0   0   0   0   0   0   0
  -1   2  -1   0   0   0   0   0   0   0   0   0   0   0   0   0
   0  -1   2  -1   0   0   0   0   0   0   0   0   0   0   0   0
   0   0  -1   2  -1   0   0   0   0   0   0   0   0   0   0   0
   0   0   0  -1   2  -1   0   0   0   0   0   0   0   0   0   0
   0   0   0   0  -1   2  -1   0   0   0   0   0   0   0   0   0
   0   0   0   0   0  -1   2  -1   0   0   0   0   0   0   0   0
   0   0   0   0   0   0  -1   2  -1   0   0   0   0   0   0   0
   0   0   0   0   0   0   0  -1   2  -1   0   0   0   0   0   0
   0   0   0   0   0   0   0   0  -1   2  -1   0   0   0   0   0
   0   0   0   0   0   0   0   0   0  -1   2  -1   0   0   0   0
   0   0   0   0   0   0   0   0   0   0  -1   2  -1   0   0   0
   0   0   0   0   0   0   0   0   0   0   0  -1   2  -1   0   0
   0   0   0   0   0   0   0   0   0   0   0   0  -1   2  -1   0
   0   0   0   0   0   0   0   0   0   0   0   0   0  -1   2  -1
   0   0   0   0   0   0   0   0   0   0   0   0   0   0  -1   2
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0  -1
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0

  Columns 17 through 20

   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
   0   0   0   0
```

Matlab Output Q4 (Exercise 1.2.20) - part (b)
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```
 1.000000000000000e+00
-6.661338147750939e-16
 4.440892098500626e-16
 1.110223024625157e-16
 3.885780586188048e-16
-8.326672684688674e-17
                     0
 2.220446049250313e-16
                     0
                     0
-8.881784197001252e-16
-1.000000000000000e+00
 6.661338147750939e-16
-4.440892098500626e-16
-1.110223024625157e-16
 2.220446049250313e-16
```

Print-out for Q5 - part (c)

[redacted]

```
function x = backsubstitute(A,b)

n = length(b);

for i = n:-1:1
    x(i,1) = b(i);

    for j = i+1:n
        x(i,1) = x(i,1) - A(i,j)*x(j);
    end

    x(i,1) = x(i,1)/A(i,i);
end

return;
```

Matlab Output Q5-part (d)

```
>> U = [-1 3 1 2; 0 -1 3 1; 0 0 -1 3; 0 0 0 -1]

U =

  -1   3   1   2
   0  -1   3   1
   0   0  -1   3
   0   0   0  -1

>> b = [-9; -3; 5; -1]

b =

  -9
  -3
   5
  -1

>> x = backsubstitute(U,b)

x =

   3
  -2
  -2
   1

>> x = U\b

x =

   3
  -2
  -2
   1
```