MATH 304

SOLUTIONS TO HOMEWORK 8

Q1.

ODE 1

(i) Euler's method

$$y_0 = -1$$

$(t_1 = 0.5)$ $y_1 = y_0 + hf(t_0, y_0)$

$$= -1 + 0.5(-1)^2 = -0.5$$

$$y_2 = y_1 + hf(t_1, y_1)$$

$$= -0.5 + 0.5(-0.5)^2 = -0.375$$

(ii) 2-step Adams-Bashforth

$$y_0 = -1$$

$y_1 = -0.5$ (By Euler's method)

$$y_2 = y_1 + h\left(\frac{3}{2}f(t_1, y_1) - \frac{1}{2}f(t_0, y_0)\right)$$

$$= -0.5 + 0.5\left(\frac{3}{2}(-0.5)^2 - \frac{1}{2}(-1)^2\right)$$

$$= -0.5 + 0.5\left(\frac{-0.25}{2}\right) = 0.5\left(-\frac{9}{8}\right)$$

$$= -4.5/8$$

(iii) Trapezoidal Method

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_1)\right)$$

$$= -1 + \frac{0.5}{2}\left((-1)^2 + y_1^2\right)$$

$$\Longrightarrow$$

$$y_1 - 0.25\,y_1^2 = -0.75$$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_2)\right)$$

$$= y_1 + 0.25\left(y_1^2 + y_2^2\right)$$

$$\Longrightarrow$$

$$y_2 - 0.25y_2^2 - y_1 - 0.25y_1^2 = 0$$

Resulting non-linear system

$$\begin{bmatrix} y_1 - 0.25y_1^2 + 0.75 \\ y_2 - 0.25y_2^2 - y_1 - 0.25y_1^2 \end{bmatrix} = 0$$

in the unknowns y_1 and y_2

ODE 2

(i) Euler's method

$$y_0 = e$$
$$y_1 = y_0 + hf(t_0, y_0)$$
$$= e + 0.5(-2e \sin 0) = e$$
$$y_2 = y_1 + hf(t_1, y_1)$$
$$= e + 0.5(-2e \sin 0.5)$$
$$= e(1 - \sin 0.5)$$

(iii) Trapezoidal Method

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_1)\right)$$
$$= e + 0.25\left(0 + (-2y_1 \sin 0.5)\right)$$
$$\Longrightarrow$$
$$y_1 + 0.5y_1 \sin 0.5 = e$$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_2)\right)$$
$$= y_1 + 0.25\left((-2y_1 \sin 0.5 - 2y_2 \sin 1)\right)$$
$$\Longrightarrow$$
$$y_2 + 0.5y_2 \sin 1 - y_1 + 0.5y_1 \sin 0.5 = 0$$

Resulting non-linear system

$$\begin{bmatrix} y_1 + 0.5y_1 \sin 0.5 - e \\ y_2 + 0.5y_2 \sin 1 - y_1 + 0.5y_1 \sin 0.5 \end{bmatrix} = 0$$

in the unknowns y_1 and y_2.

(iii) 2-step Adams – Bashforth

$$y_0 = e, \qquad y_1 = e \quad \left(\begin{array}{c}\text{Given by}\\ \text{Euler's method}\end{array}\right)$$

$$y_2 = y_1 + h\left(\tfrac{3}{2} f(t_1, y_1) - \tfrac{1}{2} f(t_0, y_0)\right)$$

$$= e + 0.5\left(\frac{3}{2}(-2e\sin 0.5) - \frac{1}{2}(0)\right)$$

$$= e\,(1 - 1.5\sin 0.5)$$

Q2.

(i)

$$\phi := \phi(f, h, y(t_k), y(t_{k+1}))$$

$$:= y(t_k) + h\overbrace{f(t_{k+1}, y(t_{k+1}))}^{y'(t_{k+1})}$$

$$= \text{(By Taylor's thm)}$$

$$y(t_k) + h\left(y'(t_k) + h y''(t_k) + \frac{h^2}{2} y'''(t_k) + O(h^3)\right)$$

$$= y(t_k) + h y'(t_k) + h^2 y''(t_k) + \frac{h^3}{2} y'''(t_k) + O(h^4)$$

On the other hand

$$y(t_{k+1}) = y(t_k) + h y'(t_k) + \frac{h^2}{2} y''(t_k) + O(h^3)$$

Therefore

$$\phi - y(t_{k+1}) = \frac{h^2}{2} y''(t_k) + O(h^3)$$

The method is of order one.

Q2

(ii)

$$\phi := \phi(f, h, y(t_k), y(t_{k+1}))$$

$$:=$$

$$y(t_k) + \frac{h}{2}\Big(\underbrace{f(t_k, y(t_k))}_{y'(t_k)} + \underbrace{f(t_{k+1}, y(t_{k+1}))}_{y'(t_{k+1})}\Big)$$

$$= \text{(By Taylor's thm)}$$

$$y(t_k) + \frac{h}{2}y'(t_k) + \frac{h}{2}\Big(y'(t_k) + hy''(t_k) + \frac{h^2}{2}y'''(t_k) + O(h^3)\Big)$$

$$=$$

$$y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + \frac{h^3}{4}y'''(t_k) + O(h^4).$$

On the other hand

$$y(t_{k+1}) = y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + \frac{h^3}{6}y'''(t_k) + O(h^4)$$

Therefore

$$\phi - y(t_{k+1}) = \frac{h^3}{12}y'''(t_k) + O(h^4).$$

The method is of order two.

Q3 (a)

The exact solution is of the form

$$v(t) = c\left(\frac{gm}{k} + e^{-\frac{kt}{m}}\right)$$

for some constant

Here k/m is close to zero and it is apparent from the figure that $v(t)$ approaches $c\left(\frac{gm}{k}\right)$ slower. It indeed resembles a linear function on $[0,3]$. The numerical solution on $[0,20]$ is also attached. Here it becomes clear that the solution is not linear

(b) When $\frac{k}{m}$ is larger as in this part, $v(t)$ quickly approaches the constant $c\left(\frac{gm}{k}\right)$. In this it is clear from the numerical solution that $v(t)$ is highly nonlinear

```
function f = gravity(x)
% Task : Computes f(t,v) = g - kdm*v

t = x(1);
v = x(2);

g = 9.81;
kdm = 0.05;

f = g - kdm*v


return
```

Figure for Q3(a)

Numerical Solution on [0,3]

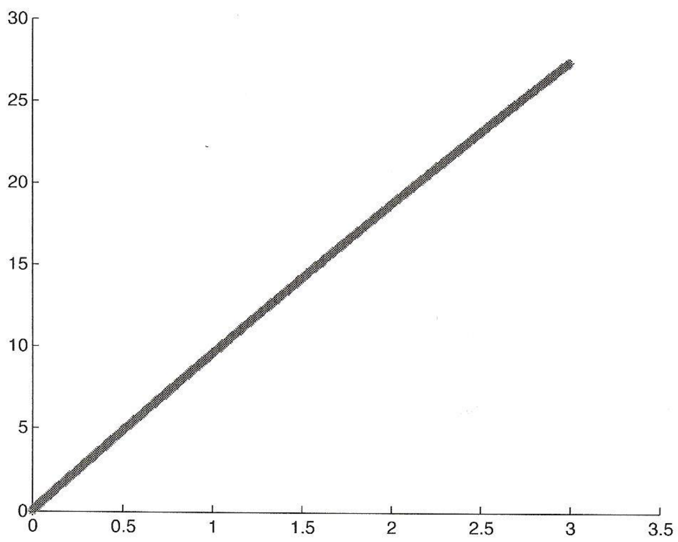


Figure for Q3(a)

Numerical Solution on [0, 20]

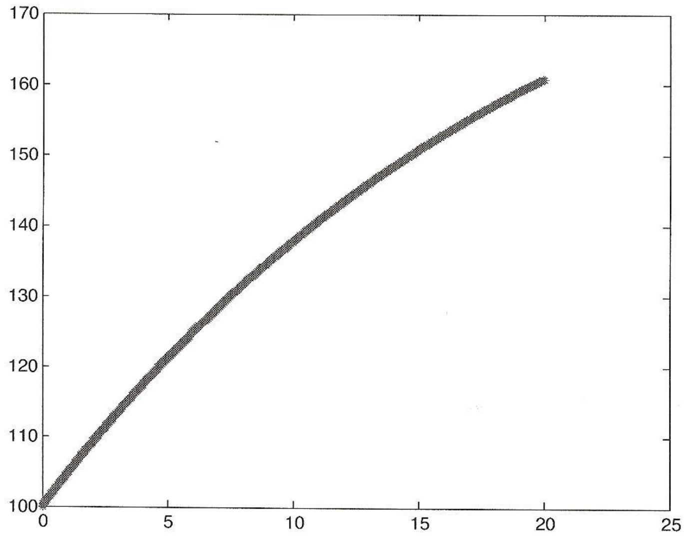
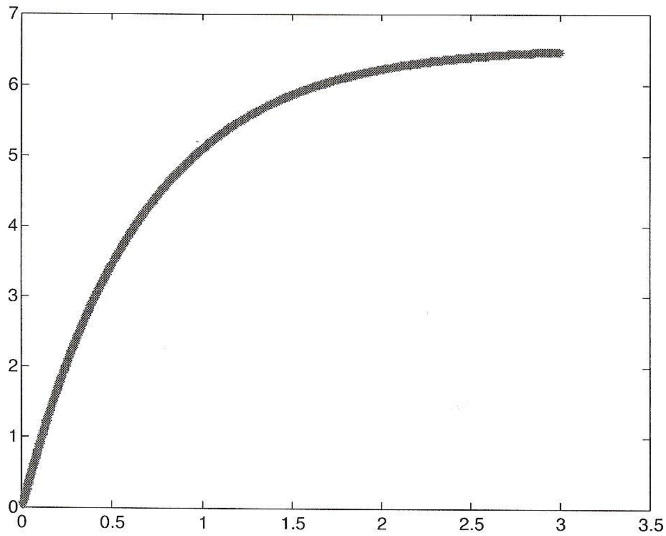


Figure for Q3(b)



Q4.

(a) For Euler's method

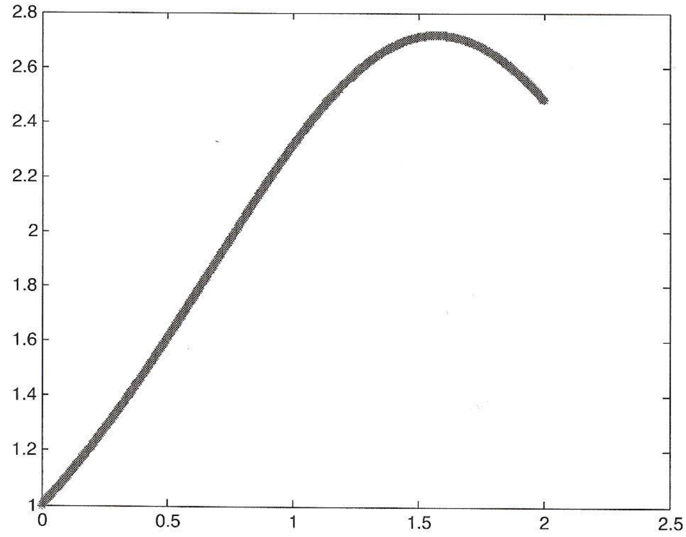
* The numerical solutions cannot be distinguished from the exact solution for $h = 10^{-2}, 10^{-3}, 10^{-4}$
* The global error decays proportional to $O(h)$.

(b) For Trapezoidal method similar remarks apply → The global error decays But proportional to h^2.

m-file for Q4

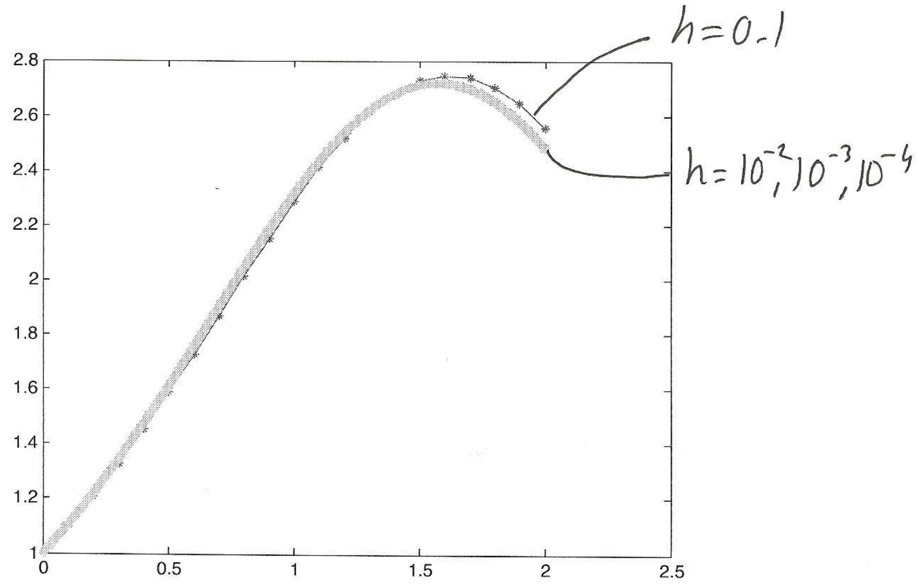

```
function f = fun_ODEQ4(x)
% Task : Computes f(t,y) = y*cos(t)

t = x(1);
y = x(2);

f = y*cos(t);

return
```

Q4
Exact Solution



Q4 (a)

Numerical Solutions



Q4(a)
Matlab Output for Euler's Method

```
>> [yreal,treal] = generate_fun('fun_real_trig',0,2,0.001);
>> plot(treal,yreal,'r-*')
>> figure(1)
>> hold on
>> [yvec,tvec] = Euler('fun_ODEQ4',0,2,0.1,1);
>> plot(tvec,yvec,'b-*')
>> err1 = yvec(21) - exp(sin(2))

err1 =

    0.0747

>>
>>
>> [yvec,tvec] = Euler('fun_ODEQ4',0,2,0.01,1);
>> plot(tvec,yvec,'g-*')
>> err2 = yvec(201) - exp(sin(2))

err2 =

    0.0075

>> [yvec,tvec] = Euler('fun_ODEQ4',0,2,0.001,1);
>> plot(tvec,yvec,'m-*')
>> err3 = yvec(2001) - exp(sin(2))

err3 =

   7.5137e-04

>> [yvec,tvec] = Euler('fun_ODEQ4',0,2,0.0001,1);
>> plot(tvec,yvec,'c-*')
>> err4 = yvec(20001) - exp(sin(2))

err4 =

   7.5141e-05
```

Q4(b)
Matlab Output for Adams-Bashforth

```
>> [yreal,treal] = generate_fun('fun_real_trig',0,2,0.001);
>> plot(treal,yreal,'r-*')
>> figure(1)
>> hold on
>> [yvec,tvec] = Adam_Bash('fun_ODEQ4',0,2,0.1,1);
>> plot(tvec,yvec,'b-*')
>> err1 = yvec(21) - exp(sin(2))

err1 =

    0.0038

>> [yvec,tvec] = Adam_Bash('fun_ODEQ4',0,2,0.01,1);
>> plot(tvec,yvec,'g-*')
>> err2 = yvec(201) - exp(sin(2))

err2 =

   3.0824e-05

>> [yvec,tvec] = Adam_Bash('fun_ODEQ4',0,2,0.001,1);
>> plot(tvec,yvec,'m-*')
>> err3 = yvec(2001) - exp(sin(2))

err3 =

   3.0159e-07

>> [yvec,tvec] = Adam_Bash('fun_ODEQ4',0,2,0.0001,1);
>> plot(tvec,yvec,'c-*')
>> err4 = yvec(20001) - exp(sin(2))

err4 =

   3.0097e-09
```

Q4(b)
Numerical Solutions

$h = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$