

Math 304 (Spring 2012)

Study Guide for Weeks 5-6

This homework concerns the following topics.

- LU Factorization
- LU Factorization with Partial Pivoting
- Least Squares Problem

Homework 3 (due on April 6th, Friday by 5pm)

Questions 3, 4 and 8 require Matlab. Attach your Matlab outputs and routines for these questions. Questions 5 and 9 are bonus questions. You will receive extra credit if you solve them fully.

1. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -2 \\ 2 \\ -7 \end{bmatrix}.$$

Solve the linear system $Ax = b$ by first computing a factorization of the form

$$PA = LU$$

where $P \in \mathbb{R}^{3 \times 3}$ is a permutation matrix, $L \in \mathbb{R}^{3 \times 3}$ is a unit lower triangular matrix (*i.e.* a lower triangular matrix with 1s on the main diagonal) and $U \in \mathbb{R}^{3 \times 3}$ is an upper triangular matrix. Perform all calculations by hand.

2. Given an invertible matrix $A \in \mathbb{R}^{n \times n}$ and a vector $y_0 \in \mathbb{R}^n$. Define the sequence of vectors $\{y_k\}$ in \mathbb{R}^n for $k \geq 1$ such that y_k satisfies

$$Ay_k = y_{k-1}.$$

Devise an algorithm for the efficient computation of y_1, y_2, \dots, y_n for a general matrix $A \in \mathbb{R}^{n \times n}$. Provide also the total flop count for your algorithm.

3. Given an invertible $n \times n$ dense matrix A (dense in this context means most of the entries, for instance more than half of the entries, of A are nonzero). The standard approach to solve the linear system $Ax = b$ is as follows.

1. Compute an LU factorization $PA = LU$ by partial pivoting.
2. Permute the right-hand side matrix, that is form $\hat{b} = Pb$.
3. Let $\hat{x} := Ux$. Solve the lower triangular system $L\hat{x} = \hat{b}$ by forward substitution.

4. Solve the upper triangular system $Ux = \hat{x}$ by back substitution.

- (a) Write a Matlab function to compute the LU factorization of A by partial pivoting. Your function must take a matrix A as input and return a lower triangular matrix L and an upper triangular matrix U as well as a vector P as outputs. The vector P represents the permutation matrix. In particular at the k th iteration if the k th and l th rows are swapped, then keep this information as $P(k) = l$. Your Matlab routine should look like

```
function [P,L,U] = lu_factor_pivot(A)

return;
```

- (b) Implement a Matlab routine to apply a given permutation matrix P to a vector b , that is implement a routine of the form

```
function [bhat] = permuteb(P,b)

return;
```

where $bhat = Pb$.

- (c) Write another Matlab function to solve the system $Ax = b$ following the steps described above. Here is how your function should look like

```
function x = linear_sys_solver(A,b)

return;
```

The routines for forward and back substitution are provided on the course website.

4. Cleve Moler's book "numerical computing with Matlab" is freely available at

<http://www.mathworks.com/moler/lu.pdf>

In particular Question 2.3 in Moler's book concerns the calculation of the net forces on a plane truss, that requires the solution of a linear system. Solve the linear system in Matlab using your linear system solver based on LU factorization from Question 3.(c).

5. (*) The k th principal submatrix of $A \in \mathbb{C}^{n \times n}$ is the upper left-most $k \times k$ portion of A . Show that A has an LU factorization of the form $A = LU$ where L is unit lower triangular with ones along the diagonal, and U is invertible and upper triangular if and only if all principal submatrices of A are invertible.

6. Consider the following four points in \mathbb{R}^2 .

$$p_1 = (-4, -1), p_2 = (-2, -5), p_3 = (-1, -1) \text{ and } p_4 = (1, 2)$$

Pose the problem of finding the line $\ell(t) = x_1 t + x_0$ minimizing

$$\sqrt{\sum_{j=1}^4 (\ell(t_j) - y_j)^2},$$

where t_j, y_j are such that $p_j = (t_j, y_j)$ for $j = 1, 2, 3, 4$, with respect to the unknowns x_0, x_1 as a least-squares problem of the form

$$\text{minimize}_x \|Ax - b\|_2.$$

7. Find $x \in \mathbb{R}^2$ such that

$$\left\| \begin{bmatrix} 1 & -3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} x - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|$$

is as small as possible by exploiting the QR factorization

$$\begin{bmatrix} 1 & -3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{6} \\ 0 & 0 \end{bmatrix}.$$

8. The following data regarding the US population is taken from the United Nations common database.

Year	Population (in millions)
1900	76.00
1910	91.97
1920	105.71
1930	123.20
1940	131.67
1950	150.70
1960	179.32
1970	203.21
1980	226.51
1990	249.63
2000	281.42

For numerical accuracy it is a good idea to make the change of variables $s = \frac{t-1950}{50}$ where t denotes the year.

Model the US population by a cubic function of the form $p(s) = x_3s^3 + x_2s^2 + x_1s + x_0$. Determine the cubic model that solves the least squares problem for the US population based on the available data above. Use the built-in Matlab routine to solve the least squares problem. Plot the best cubic model for the least squares problem as well as the data points on the same graph in Matlab. Estimate the population in the US in 2020 using the best cubic model.

9. (*) Let $A \in \mathbb{R}^{m \times n}$ with $m > n$ and $b \in \mathbb{R}^m$. Show that the x value making $\|Ax - b\|$ as small as possible is unique if and only if $\dim(\text{Col}(A)) = n$. (Hint: Every $A \in \mathbb{R}^{m \times n}$ has a QR factorization.)