SOLUTIONS TO MIDTERM I

Q1

(a) Newton update rule

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

$$= x_k - \frac{(a - 1/x_k)}{(1/x_k^2)}$$

$$= x_k - (a x_k^2 - x_k)$$

$$= 2x_k - a x_k^2$$

(b)

$$\lim_{k\to\infty} \frac{|x_{k+1} - 1/a|}{|x_k - 1/a|^2} = \lim_{k\to\infty} \frac{|2x_k - a x_k^2 - 1/a|}{|x_k - 1/a|^2}$$

$$= \lim_{k\to\infty} \frac{|a|\,|x_k - 1/a|^2}{|x_k - 1/a|^2}$$

$$= |a|$$

so the rate of convergence is at least quadratic.

On the other hand

$$\lim_{k\to\infty} \frac{|x_{k+1} - 1/a|}{|x_k - 1/a|^3} = \lim_{k\to\infty} \frac{|a|}{|x_k - 1/a|}$$

$$= \infty$$

so the rate of convergence is not cubic.

Q2

A₁ is positive definite

$$x^T \underbrace{A_1}_{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} x = 2x_1^2 + x_2^2 > 0$$

for all $x \neq 0$.

Recall also for any invertible X and positive definite A

$$XAX^T \succ 0$$

Consequently

$$A_1 = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}}_{X^T} \succ 0$$

since $\begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}$ is invertible.

Cholesky factorization of A_1

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & 3 \end{bmatrix}}_{R^T} \underbrace{\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 3 \end{bmatrix}}_{R}$$

A_2 is not positive definite

This is due to the fact that diagonal elements of A_2 are negative. For instance

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & -3 \end{bmatrix}}_{A_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -3 < 0$$

Q3

(a) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{r_2 := r_2 - r_1 \\ r_3 := r_3 - r_1}]{} \begin{bmatrix} 1 & -1 & -1 \\ \cancel{0}_1 & 2 & -1 \\ \cancel{0}_1 & 2 & 0 \end{bmatrix}$

$\xrightarrow[r_3 := r_3 - r_2]{} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & \cancel{0}1 & 1 \end{bmatrix}$

LU Factorization

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

(b) $Ax = b \iff L\underbrace{Ux}_{\hat{x}} = b$

Forward Stage

Solve $L\hat{x} = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

Backward Stage

Solve $Ux = \hat{x}$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Q4

(1) R is invertible $\Longrightarrow A \succ 0$

Let $x \neq 0$, and assume R is invertible.

$$x^T A x = x^T R^T R x$$

$$= y^T y \quad \left(\text{where } y = Rx \neq 0 \text{ since } R \text{ is invertible}\right)$$

$$= \|y\|^2 > 0$$

Consequently $A \succ 0$.

(2) $A \succ 0 \Longrightarrow R$ is invertible

equivalently

R is not invertible $\Longrightarrow A \nsucc 0$

Since R is not invertible, there exists an $x \neq 0$ such that

$$Rx = 0.$$

Consequently

$$\begin{aligned} x^T A x &= x^T R^T R x \\ &= \cancel{(x^T} (Rx)^T (Rx) \\ &= 0^T 0 = 0 \end{aligned}$$

so $A \not\succ 0$.

Q5

There exist a δ such that

$$|g'(x)| < 1$$

for all $x \in [p-\delta, p+\delta]$ due to continuity of g'.

For any $x \in [p-\delta, p+\delta]$

$$\begin{aligned} |g(x) - g(p)| &= |g'(\tilde{x})| |x-p| \quad \text{(By mean value thm)} \\ &< |x-p| \leq \delta \end{aligned}$$

where $\tilde{x} \in (p-\delta, p+\delta)$. This means that

$$|g(x)-g(p)| = |g(x)-p| < \delta$$

$$\Longrightarrow$$

$$p-\delta < g(x) < p+\delta.$$

We deduced that

(i) $|g'(x)| < 1$ for all $x \in [p-\delta, p+\delta]$, and

(ii) $g(x) \in [p-\delta, p+\delta]$ for all $x \in [p-\delta, p+\delta]$.

As proven in class these two conditions guarantee the existence of a unique fixed point in $[p-\delta, p+\delta]$. Furthermore, provided $p_0 \in [p-\delta, p+\delta]$, these two conditions guarantee that

$$\lim_{k\to\infty} p_k = p.$$