SOLUTIONS TO MIDTERM 2

Q1.

(a)

minimize

$$\sqrt{(x_1+x_0-5)^2+(6x_1+x_0-12)^2+(9x_1+x_0-13)^2}$$

$$=$$

$$\left\| \begin{bmatrix} x_1+x_0-5 \\ 6x_1+x_0-12 \\ 9x_1+x_0-13 \end{bmatrix} \right\|_2$$

$$=$$

$$\left\| \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 6 \\ 1 & 9 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}}_{x} - \underbrace{\begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}}_{b} \right\|_2$$

(b)

$$Q^T b = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 32/\sqrt{3} \\ 8/\sqrt{2} \\ 2/\sqrt{6} \end{bmatrix}$$

Need to solve

$$\begin{bmatrix} \sqrt{3} & 5\sqrt{3} \\ 0 & 3\sqrt{2} \end{bmatrix} x = \begin{bmatrix} 32/\sqrt{3} \\ 8/\sqrt{2} \end{bmatrix}$$

$$x_2 = \frac{8/\sqrt{2}}{3\sqrt{2}} = \frac{4}{3}$$

$$\cancel{\sqrt{3}}x_1 + 5\cancel{\sqrt{3}} \cdot \frac{4}{3} = \frac{32}{\cancel{\sqrt{3}}\,3} \implies x_1 = \cancel{\tfrac{12}{3}}\,4$$

Q2.

Need to solve

$$(*)\quad Ax_1 = e_1,\ Ax_2 = e_2,\ \dots,\ Ax_n = e_n$$

where

$$A^{-1} = [x_1\ x_2\ \dots\ x_n]$$

One LU factorization is sufficient for systems in (*), since they have the same coefficient matrix

<u>Algorithm</u> (Inverse of a Matrix)

Input: $A \in \mathbb{R}^{n\times n}$

Output: $X \in \mathbb{R}^{n\times n}$ such that $X = A^{-1}$.

Compute an LU factorization of the form

$$\underbrace{P}_{\text{permutation matrix}}A = \underbrace{L}_{\text{lower-triangular}}\,U \text{ — upper triangular} \qquad \boxed{\frac{2n^3}{3}\text{ FLOPS}}$$

for $j = 1, \dots, n$

$\hat{b} = P^T e_j$ $\boxed{\text{NO FLOPS JUST REORDERING OF ENTRIES OF } e_j}$

Solve $L\hat{x} = \hat{b}$ by forward substitution. $\boxed{n^2 \text{ FLOPS}}$

Solve $Ux = \hat{x}$ by back substitution $\boxed{n^2 \text{ FLOPS}}$

$X_j \leftarrow x$

end

return X

$$\text{TOTAL \# FLOPS} = \frac{2n^3}{3} + n\,(2n^2) + O(n^2)$$

$$= 8n^3/3 + \underbrace{O(n^2)}_{\text{Terms proportional to } n^2 \text{ or smaller}}$$

Q3.

Use the lagrange polynomial

$$p_1(x) = f(x_0)\,l_0(x) + f(x_1)\,l_1(x)$$

$$= f(1)\,\frac{(x-3)}{(1-3)} + f(3)\,\frac{(x-1)}{3-1}$$

$$= -\frac{f(1)}{2}(x-3) + \frac{f(3)}{2}(x-1)$$

$$\int_0^4 f(x)\,dx \approx \int_0^4 p_1(x)\,dx$$

$$= -\frac{f(1)}{2}\int_0^4 (x-3)\,dx + \frac{f(3)}{2}\int_0^4 (x-1)\,dx$$

$$= 2f(1) + 2f(3)$$

Q4.

$$q_2(x) \perp 1$$

$$\int_0^1 x^2 + \alpha_1 x + \alpha_0 \, dx = 0$$

$$\Longrightarrow \left.\frac{x^3}{3} + \alpha_1 \frac{x^2}{2} + \alpha_0 x \right|_0^1 = \frac{1}{3} + \frac{\alpha_1}{2} + \alpha_0 = 0$$

$$q_2(x) \perp x$$

$$\int_0^1 (x^2 + \alpha_1 x + \alpha_0)\, x \, dx = 0$$

$$\Longrightarrow \left.\frac{x^4}{4} + \alpha_1 \frac{x^3}{3} + \alpha_0 \frac{x^2}{2} \right|_0^1 = \frac{1}{4} + \frac{\alpha_1}{3} + \frac{\alpha_0}{2} = 0$$

Solve

$$\alpha_0 + \frac{\alpha_1}{2} = \frac{-1}{3}$$
$$\frac{\alpha_0}{2} + \frac{\alpha_1}{3} = \frac{-1}{4} \quad \Longrightarrow \quad \alpha_0 = +1/6, \quad \alpha_1 = -1$$

Orthogonal polynomial

$$q_2(x) = x^2 - x + 1/6$$

Q5.

(a)

$$F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

$$F(\underbrace{\mu}_{\mathbb{R}}, \underbrace{v}_{\mathbb{R}^n}) = \begin{bmatrix} (A + \mu B) v \\ v^T v - 1 \end{bmatrix}$$

We are seeking to solve

$$F(\mu, v) = \begin{bmatrix} (A + \mu B) v \\ v^T v - 1 \end{bmatrix} = 0$$

(b) Newton iteration

let $x_k := (\mu_k, v_k)$

Jacobian $F'(\mu, v) = \begin{bmatrix} Bv & A + \mu B \\ 0 & 2v^T \end{bmatrix}$

Let p_k satisfy

$$\underbrace{\begin{bmatrix} Bv_k & A+\mu_k B \\ 0 & 2v_k^T \end{bmatrix}}_{F'(\mu_k, v_k)} p_k = -\underbrace{\begin{bmatrix} (A+\mu_k B)v_k \\ v_k^T v_k - 1 \end{bmatrix}}_{F(\mu_k, v_k)}$$

Then

$$x_{k+1} := (\mu_{k+1}, v_{k+1}) = (\mu_k, v_k) + p_k$$