Contraction Mapping Thm

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Theorem 0.1 (Contraction Mapping Thm)**.** *Let D be a closed subset of $\mathbb{R}^n$, and $g : D \to \mathbb{R}^n$ satisfy the following:*

- $g(D) \subseteq D$.
- *There exists $\gamma \in (0, 1)$ such that*

$$\|g(x) - g(y)\|_\infty \quad \leq \quad \gamma\|x - y\|_\infty \quad \forall x, y \in D.$$

Theorem 0.2 (Contraction Mapping Thm)**.** *Let D be a closed subset of $\mathbb{R}^n$, and $g : D \to \mathbb{R}^n$ satisfy the following:*

- $g(D) \subseteq D$.
- *There exists $\gamma \in (0, 1)$ such that*

$$\|g(x) - g(y)\|_\infty \quad \leq \quad \gamma \|x - y\|_\infty \quad \forall x, y \in D.$$

Then

(1) *there exists a unique $x_* \in D$ such that $g(x_*) = x_*$,*

(2) $\{x^{(k)}\},\ x^{(k+1)} = g(x^{(k)})$ *converges to x_* for all $x^{(0)} \in D$.*

Jacobi iteration to solve $Ax = b$ (given $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$)

Based on $\quad Ax = b \quad \iff \quad x = -D^{-1}(L+U)x + D^{-1}b$

Jacobi iteration to solve $Ax = b$ (given $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$)

Based on $\quad Ax = b \quad \Longleftrightarrow \quad x = -D^{-1}(L + U)x + D^{-1}b$

$$\mathbf{x^{(k+1)}} \quad = \quad \mathbf{-D^{-1}(L + U)x^{(k)} + D^{-1}b}$$

Jacobi iteration to solve $Ax = b$

$$g(x) = -D^{-1}(L+U)x + D^{-1}b$$

$$\begin{aligned} \|g(x) - g(y)\|_\infty &= \|D^{-1}(L+U)(x-y)\|_\infty \\ &\leq \|D^{-1}(L+U)\|_\infty \|x-y\|_\infty \end{aligned}$$

Jacobi iteration to solve $Ax = b$

$$g(x) = -D^{-1}(L+U)x + D^{-1}b$$

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If $|a_{jj}| > \sum_{k=1, k\neq j}^{n} |a_{jk}|, \ j = 1, \ldots, n,$ then $\|D^{-1}(L+U)\|_\infty < 1$.

(1) $g(\overline{B}(x_*, \varepsilon)) \subseteq \overline{B}(x_*, \varepsilon)$ for any $\varepsilon > 0$.

Jacobi iteration to solve $Ax = b$

$$\begin{aligned} g(x) &= -D^{-1}(L+U)x + D^{-1}b \end{aligned}$$

$$\begin{aligned} \|g(x) - g(y)\|_\infty &= \|D^{-1}(L+U)(x-y)\|_\infty \\ &\leq \|D^{-1}(L+U)\|_\infty \|x-y\|_\infty \end{aligned}$$

If $|a_{jj}| \; > \displaystyle\sum_{k=1, k\neq j}^{n} |a_{jk}|, \; j = 1, \ldots, n,$ then $\|D^{-1}(L+U)\|_\infty < 1$.

(2) g is a contraction on $g(\overline{B}(x_*, \varepsilon))$ with $\gamma = \|D^{-1}(L+U)\|_\infty$.

Jacobi iteration to solve $Ax = b$

$$\mathbf{x^{(k+1)}} \;=\; -\mathbf{D^{-1}}(\mathbf{L}+\mathbf{U})\mathbf{x^{(k)}} + \mathbf{D^{-1}b}$$

If $|a_{jj}| \; > \sum_{k=1, k\neq j}^{n} |a_{jk}|, \; j = 1, \ldots, n$, then

$\{x^{(k)}\}$ converges to x_* for all $x^{(0)}$.