Check for the Convergence of the Simultaneous Iteration

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Suppose $g : \mathbb{R}^n \to \mathbb{R}^n$ has continuous first derivatives, that is

$$\frac{\partial g_i(x)}{\partial x_j} \quad i, j = 1, \ldots, n$$

are continuous. Furthermore,

$$\|J_g(x_*)\|_\infty \ < \ 1.$$

The sequence $\{x^{(k)}\},\ x^{(k+1)} = g(x^{(k)})$ converges to x_* for all $x^{(0)}$ sufficiently close to x_*.

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x) \;=\; \begin{bmatrix} x_1 - x_2 e^{x_1 x_2} \\ x_1^2 + x_1 x_2 + x_2 \end{bmatrix}$$

$$J_g(x) \;=\; \begin{bmatrix} 1 - x_2^2 e^{x_1 x_2} & 1 - e^{x_1 x_2} - x_1 x_2 e^{x_1 x_2} \\ 2x_1 + x_2 & x_1 + 1 \end{bmatrix}$$

Given $f : \mathbb{R} \to \mathbb{R}$, $x, p \in \mathbb{R}$, define

$$\phi(\alpha) \ := \ f(x + \alpha p).$$

Fundamental theorem

$$\phi(1) = \phi(0) + \int_0^1 \phi'(t)\, dt$$

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Taylor's theorem with integral remainder

$$\mathbf{f(x + p) = f(x) + \int_0^1 f'(x + tp)p\, dt}$$

Given $f : \mathbb{R}^n \to \mathbb{R}^n$, $x, p \in \mathbb{R}^n$, define

$$\phi_j(\alpha) \ := \ f_j(x + \alpha p).$$

Fundamental theorem

$$\phi_j(1) = \phi_j(0) + \int_0^1 \phi_j'(t)\, dt$$

Equivalently,

$$f_j(x + p) = f_j(x) + \int_0^1 \nabla f_j(x + tp)^T p\, dt$$

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Equivalently,

$$\begin{bmatrix} f_1(x+p) \\ f_2(x+p) \\ \vdots \\ f_n(x+p) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} + \int_0^1 \begin{bmatrix} \nabla f_1(x+tp)^T p \\ \nabla f_2(x+tp)^T p \\ \vdots \\ \nabla f_n(x+tp)^T p \end{bmatrix} dt$$

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Taylor's theorem with integral remainder

$$\mathbf{f}(\mathbf{x} + \mathbf{p}) = \mathbf{f}(\mathbf{x}) + \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{J_f}(\mathbf{x} + \mathbf{tp})\mathbf{p}\, \mathbf{dt}$$