Background on Eigenvalues

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A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbb{R}^{n\times n}$ if

$$Av = \lambda v \quad \exists v \in \mathbb{C}^n, \;\; v \neq 0. \tag{0.1}$$

- Nonzero v in (0.1) is an eigenvector corresponding to λ.

Hand calculation

$$\begin{aligned} Av = \lambda v \; \exists v \neq 0 \quad &\Longleftrightarrow \quad (A - \lambda I)v = 0 \; \exists v \neq 0 \\ &\Longleftrightarrow \quad \underbrace{\det(A - \lambda I)}_{p(\lambda)} = 0 \end{aligned}$$

- $p(\lambda)$ is the characteristic polynomial of A.

Example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right)$$
$$= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$

Eigenvector corresponding to $\lambda_1 = 4$

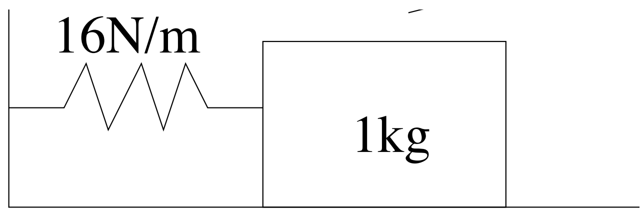
$$(A - 4I)v_1 = 0, \qquad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0$$

Hence, $v_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any nonzero scalar c_1.

Eigenvector corresponding to $\lambda_2 = 2$

$$(A - 2I)v_2 = 0, \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_2 = 0$$

$v_2 = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for any nonzero scalar c_2.



A Motivating Example

$x(t)$ is the displacement.

Friction in the ground, a force of $-kx'(t)$,
where k is the friction constant.

$$x''(t) \;=\; -kx'(t) - 16x(t)$$

Letting $y_1(t) := x(t), \quad y_2(t) := x'(t)$

$$\begin{aligned} y_2'(t) &= -ky_2(t) - 16y_1(t) \\ y_1'(t) &= y_2(t) \end{aligned}$$

that is

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

$$y'(t) = \underbrace{A}_{n\times n} y(t)$$

Eigenvalues of A: $\lambda_1, \ldots, \lambda_n$
Corresponding eigenvectors: $v_1, \ldots, v_n$

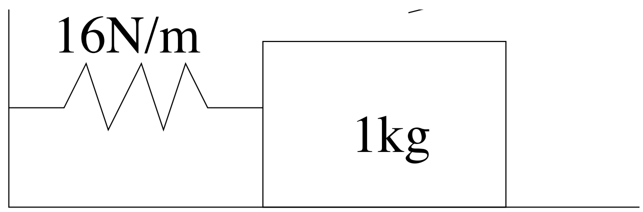
Solution

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

where $c_1, \ldots, c_n$ are constants to be determined using $y(0)$.

Verify

$$\begin{aligned}(c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n)' &= (c_1 \lambda_1 e^{\lambda_1 t} v_1 + \cdots + c_n \lambda_n e^{\lambda_n t} v_n) \\ &= (c_1 e^{\lambda_1 t} A v_1 + \cdots + c_n e^{\lambda_n t} A v_n) \\ &= A(c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n)\end{aligned}$$



$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Characteristic polynomial:
$p(\lambda) = (-\lambda)(-k - \lambda) + 16 = \lambda^2 + k\lambda + 16$

Eigenvalues: $\lambda_1 = \frac{-k+\sqrt{k^2-64}}{2}, \quad \lambda_2 = \frac{-k-\sqrt{k^2-64}}{2}$

Solution:

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

where c_1, c_2 satisfies

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 v_1 + c_2 v_2 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Symmetric Eigenvalue Problem

If $A \in \mathbb{R}^{n \times n}$ is symmetric, that is $A^T = A$,

- Eigenvalues $\lambda_1, \ldots, \lambda_n$ are real numbers,
- The corresponding eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^n$ can be chosen s.t. $\{v_1, \ldots, v_n\}$ is orthonormal.