Similarity Transformations and Power Iteration

November 1, 2018

Let $A \in \mathbb{R}^{n\times n}$. A similarity transformation, for a given invertible $S \in \mathbb{R}^{n\times n}$, is of the form

$$A \;\mapsto\; S^{-1}AS \;=:\; B$$

- A, B are said to be similar matrices.

$$\det(S^{-1}AS - \lambda I) \quad = \quad \det(S^{-1}(A - \lambda I)S)$$

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- A and B have the same characteristic polynomial.
- A and B have the same eigenvalues.

Trace.

For $A \in \mathbb{R}^{n \times n}$

$$\text{Trace}(A) = \sum_{j=1}^{n} a_{jj}.$$

- $\text{Trace}(A)$ is the sum of the eigenvalues of A.
- If $A, B \in \mathbb{R}^{n \times n}$ are similar, then

$$\text{Trace}(A) = \text{Trace}(B).$$

Power Iteration.

Given $A \in \mathbb{R}^{n \times n},\ A^T = A$,
generates a sequence $\{q^{(k)}\}$ in $\mathbb{R}^n$ such that

$$q^{(k+1)} \quad := \frac{Aq^{(k)}}{\|Aq^{(k)}\|_2}$$

for a given $q^{(0)} \in \mathbb{R}^n$.

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$$q^{(k)} \quad = \quad s_k A^k q^{(0)}, \quad s_k \ := \ 1/\|A^k q^{(0)}\|_2$$

Notation

- Eigenvalues of A: $\lambda_1, \lambda_2 \ldots, \lambda_{n-1}, \lambda_n \in \mathbb{R}$ s.t.

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{n-1}| \leq |\lambda_n|.$$

- Corresponding orthonormal eigenvectors: $v_1, v_2, \ldots, v_{n-1}, v_n$

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$$\begin{aligned} q^{(0)} &= \alpha_n v_n + \alpha_{n-1} v_{n-1} + \cdots + \alpha_1 v_1 \\ Aq^{(0)} &= \alpha_n \lambda_n v_n + \alpha_{n-1} \lambda_{n-1} v_{n-1} + \cdots + \alpha_1 \lambda_1 v_1 \end{aligned}$$

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$$\begin{aligned} q^{(0)} &= \alpha_n v_n + \alpha_{n-1} v_{n-1} + \cdots + \alpha_1 v_1 \\ A q^{(0)} &= \alpha_n \lambda_n v_n + \alpha_{n-1} \lambda_{n-1} v_{n-1} + \cdots + \alpha_1 \lambda_1 v_1 \\ &\vdots \\ A^k q^{(0)} &= \alpha_n \lambda_n^k v_n + \alpha_{n-1} \lambda_{n-1}^k v_{n-1} + \cdots + \alpha_1 \lambda_1^k v_1 \end{aligned}$$