Reduction into Tridiagonal Form (Reminders from Lecture 15)

Task to do

$A \in \mathbb{R}^{n \times n}$, $A^T = A$

Find orthogonal $Q_1, \ldots, Q_{n-2} \in \mathbb{R}^{n \times n}$ such that

$$Q_{n-2}^T \ldots, Q_1^T A Q_1 \ldots Q_{n-2} \quad = \quad T$$

where $T \in \mathbb{R}^{n \times n}$ is tridiagonal.

Task to do

$$\underbrace{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}_{A}$$

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$$\underbrace{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}_{A} \mapsto \underbrace{\begin{bmatrix} x & x & 0 & 0 \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}}_{Q_1^T A Q_1}$$

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$$\underbrace{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}_{A} \mapsto \underbrace{\begin{bmatrix} x & x & 0 & 0 \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}}_{Q_1^T A Q_1} \mapsto \underbrace{\begin{bmatrix} x & x & 0 & 0 \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}}_{Q_2^T Q_1^T A Q_1 Q_2}$$

Householder Reflectors

For a number of years it seemed that the Givens' process was likely to prove the most efficient method of reducing a matrix to tri-diagonal form, but in 1958 Householder suggested that this reduction could be performed more efficiently using the elementary Hermitian matrices rather than plane rotations.

The Algebraic Eigenvalue Problem (1965), J. H. Wilkinson

Householder Reflectors

- The key to this reduction is, for a given $v \in \mathbb{R}^n$, finding an orthogonal $Q \in \mathbb{R}^{n \times n}$ such that

$$Qv = \begin{bmatrix} \pm\|v\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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- More generally, given $a, b \in \mathbb{R}^n$ satisfying $\|a\|_2 = \|b\|_2$, find orthogonal $Q \in \mathbb{R}^{n \times n}$ such that

$$Qa = b.$$

Orthogonal Projection

Let S be a subspace of $\mathbb{R}^n$. Every vector $v \in \mathbb{R}^n$ can be decomposed into

$$v \;=\; v_s + v_{s^\perp} \qquad \exists v_s \in S,\ \exists v_{s^\perp} \in S^\perp$$

in a unique way.

v_s is called the orthogonal projection of v onto S.

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$$v_s = d\left(\frac{d^T v}{d^T d}\right) = \left(\frac{dd^T}{d^T d}\right) v$$