The QR Algorithm

The Algorithm

$T^{(0)} \leftarrow T$, $k \leftarrow 0$
for $k = 0, 1, 2, \ldots$ **do**
  Choose a shift μ_{k+1}.
  Compute a QR factorization

$$T^{(k)} - \mu_{k+1} I_n \;=\; \widehat{Q}^{(k+1)} \widehat{R}^{(k+1)}.$$

  $T^{(k+1)} \leftarrow \widehat{R}^{(k+1)} \widehat{Q}^{(k+1)} + \mu_{k+1} I_n$
end for

- $T^{(k)}, T^{(k+1)}$ are orthogonally similar for all k

Preserving Tridiagonal Structure

Given $T^{(k)}$ is tridiagonal, so is $T^{(k+1)}$.

- In the QR factorization

$$T^{(k)} - \mu_{k+1} I_n \;=\; \widehat{Q}^{(k+1)} \widehat{R}^{(k+1)},$$

$\widehat{R}^{(k+1)}$ is bidiagonal.

Preserving Tridiagonal Structure

Given $T^{(k)}$ is tridiagonal, so is $T^{(k+1)}$.

- In the QR factorization

$$T^{(k)} - \mu_{k+1} I_n \; = \; \widehat{Q}^{(k+1)} \widehat{R}^{(k+1)},$$

  $\widehat{R}^{(k+1)}$ is bidiagonal.

- $\left[\widehat{R}^{(k+1)}\right]^{-1}$ is also bidiagonal.

Preserving Tridiagonal Structure

Given $T^{(k)}$ is tridiagonal, so is $T^{(k+1)}$.

- In the QR factorization

$$T^{(k)} - \mu_{k+1} I_n \;=\; \widehat{Q}^{(k+1)} \widehat{R}^{(k+1)},$$

  $\widehat{R}^{(k+1)}$ is bidiagonal.

- $\left[\widehat{R}^{(k+1)}\right]^{-1}$ is also bidiagonal.

- $\widehat{Q}^{(k+1)} = (T^{(k)} - \mu_{k+1} I_n) \left[\widehat{R}^{(k+1)}\right]^{-1}$ is tridiagonal.

Preserving Tridiagonal Structure

Given $T^{(k)}$ is tridiagonal, so is $T^{(k+1)}$.

- In the QR factorization

$$T^{(k)} - \mu_{k+1} I_n \; = \; \widehat{Q}^{(k+1)} \widehat{R}^{(k+1)},$$

  $\widehat{R}^{(k+1)}$ is bidiagonal.

- $\left[\widehat{R}^{(k+1)}\right]^{-1}$ is also bidiagonal.

- $\widehat{Q}^{(k+1)} = (T^{(k)} - \mu_{k+1} I_n) \left[\widehat{R}^{(k+1)}\right]^{-1}$ is tridiagonal.

- $T^{(k+1)} = \widehat{R}^{(k+1)} \widehat{Q}^{(k+1)} + \mu_{k+1} I_n$ is tridiagonal.

Choice of the Shift

Rayleigh shift

$$\mu_{k+1} = t_{nn}^{(k)}$$

Wilkinson shift

$$\mu_{k+1} = \text{eigenvalue of } T^{(k)}(n-1:n\,,\; n-1:n) \text{ closest to } t_{nn}^{(k)}$$