Polynomial Interpolation

November 19, 2018

Lagrange Interpolation
Let $x_0, \ldots, x_n \in \mathbb{R}$ be distinct, and $y_0, \ldots, y_n \in \mathbb{R}$.

There exists a unique $p_n \in \mathcal{P}_n$ s.t. $p_n(x_k) = y_k$ for $k = 0, \ldots, n$.

$$\begin{aligned} p_n(x) &= L_0(x)y_0 + L_1(x)y_1 + \cdots + L_n(x)y_n, \\ L_k(x) &= \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} \end{aligned}$$

Lagrange interpolation polynomial for $f : \mathbb{R} \to \mathbb{R}$
with interpolation points $x_0, \ldots, x_n$

$$
\begin{aligned}
p_n(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_n(x)f(x_n), \\
L_k(x) &= \prod_{j=0, j \neq k}^{n} \frac{x - x_j}{x_k - x_j}
\end{aligned}
$$

(Note: $p(x_k) = f(x_k)$ for $k = 0, \ldots, n$)

Lagrange interpolation polynomial for $f : \mathbb{R} \to \mathbb{R}$
with interpolation points $x_0, \ldots, x_n$

$$\begin{aligned} p_n(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_n(x)f(x_n), \\ L_k(x) &= \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} \end{aligned}$$

(Note: $p(x_k) = f(x_k)$ for $k = 0, \ldots, n$)

Interpolation Error

$x_0, x_1, \ldots, x_n \in [a, b]$, f is $(n + 1)$ times differentiable on $[a, b]$.
For all $x \in [a, b]$, we have

$$\begin{aligned} f(x) - p_n(x) &= \frac{f^{(n+1)}(\varepsilon)}{(n + 1)!}\pi_{n+1}(x), \quad \exists \varepsilon \in (a, b) \\ \pi_{n+1}(x) &:= (x - x_0)(x - x_1) \ldots (x - x_n) \end{aligned}$$

Hermite Interpolation

Let $x_0, \ldots, x_n \in \mathbb{R}$ be distinct, and $y_0, \ldots, y_n, z_0, \ldots, z_n \in \mathbb{R}$.

There exists a $p_{2n+1} \in \mathcal{P}_{2n+1}$ s.t. $p_{2n+1}(x_k) = y_k$, $p'_{2n+1}(x_k) = z_k$ for $k = 0, \ldots, n$.

$$\begin{aligned} p_{2n+1}(x) &= \{H_0(x)y_0 + K_0(x)z_0\} + \{H_1(x)y_1 + K_1(x)z_1\} + \\ &\qquad \cdots + \{H_n(x)y_n + K_n(x)z_n\}, \end{aligned}$$

$$\begin{aligned} H_k(x) &= [L_k(x)]^2 \left[1 - 2L'(x_k)(x - x_k)\right], \\ K_k(x) &= [L_k(x)]^2 (x - x_k) \end{aligned}$$