Numerical Integration

The General Framework

$$
\begin{aligned}
\int_a^b f(x)dx &\approx \int_a^b p_n(x)dx \\
&= \int_a^b \sum_{k=0}^{n} L_k(x) f(x_k) dx
\end{aligned}
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This yields the quadrature formula

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} \underbrace{\left\{ \int_a^b L_k(x)dx \right\}}_{w_k} f(x_k)$$

Newton-Cotes Formulas

Employs equally spaced quadrature points $x_0, \ldots, x_n$ on $[a, b]$.

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$x_0 = a$, $x_1 = b$

$$\int_a^b f(x)dx \approx \left\{ \int_a^b \frac{x-b}{a-b} dx \right\} f(a) + \left\{ \int_a^b \frac{x-a}{b-a} dx \right\} f(b)$$

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$$\int_a^b f(x)dx \quad\approx\quad \frac{b-a}{2}f(a) \;+\; \frac{b-a}{2}f(b)$$

Newton-Cotes Formulas

Simpson's Rule (Newton-Cotes, $n = 2$)
$x_0 = a$, $x_1 = (a+b)/2$, $x_2 = b$

$$
\begin{aligned}
\int_a^b f(x)dx \;\approx\; & \left\{ \int_a^b \frac{(x - \frac{(a+b)}{2})(x-b)}{(a - \frac{a+b}{2}))(a-b)} dx \right\} f(a) + \\
& \left\{ \int_a^b \frac{(x-a)(x-b)}{(\frac{a+b}{2} - a)(\frac{a+b}{2} - b)} dx \right\} f\left(\frac{a+b}{2}\right) + \\
& \left\{ \int_a^b \frac{(x-a)(x - \frac{(a+b)}{2})}{(b-a)(b - \frac{a+b}{2}))} dx \right\} f(b)
\end{aligned}
$$

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& \left\{ \int_a^b \frac{(x - a)(x - b)}{(\frac{a+b}{2} - a)(\frac{a+b}{2} - b)} dx \right\} f\left(\frac{a+b}{2}\right) + \\
& \left\{ \int_a^b \frac{(x - a)(x - \frac{(a+b)}{2})}{(b - a)(b - \frac{a+b}{2}))} dx \right\} f(b)
\end{aligned}
$$

$$
\int_a^b f(x)dx \;\approx\; \frac{b-a}{6} f(a) + \frac{4(b-a)}{6} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)
$$