Best Polynomial Approximation

Problem

Best Polynomial Approximation
Given $n \in \mathbb{N}$ and $f \in C[a, b]$, find $p_n \in \mathcal{P}_n$ such that

$$\|f - p_n\|$$

is as small as possible.

C[a,b] - vector space of continuous functions
$f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$.

Norms on $C[a, b]$

A norm $\|\cdot\|$ on $C[a, b]$ satisfies

(1) $\|f\| > 0 \quad \forall f \in C[a, b]$ unless $f(x) \equiv 0$,

(2) $\|\alpha f\| = |\alpha|\|f\| \quad \forall f \in C[a, b],\ \forall \alpha \in \mathbb{R}$,

(3) $\|f + g\| \leq \|f\| + \|g\| \quad \forall f, g \in C[a, b]$.

Choice of the Norm

We will consider $\|\cdot\| = \|\cdot\|_2$ or $\|\cdot\| = \|\cdot\|_\infty$.

- ∞-norm

$$\|f\|_\infty := \max_{x \in [a,b]} |f(x)|$$

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$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$$

- 2-norm (with respect to a positive continuous weight function $w(x)$)

$$\|f\|_2 := \sqrt{\int_a^b w(x) f(x)^2 dx}$$

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$f(x) = \sqrt{x}$ on $[0, 1]$

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$$= \sqrt{(-\sqrt{1 - u}) \Big|_0^1} = 1$$