Theorem of Alternation

Definition

The Minimax Polynomial for f of Degree n
Given $n \in \mathbb{N}$ and $f \in C[a, b]$,
the polynomial $p_* \in \mathcal{P}_n$ such that

$$\|f - p_*\|_\infty = \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$$

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$$\begin{aligned} \|f - p_*\|_\infty &= \min_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty \\ &= \min_{p_n \in \mathcal{P}_n} \max_{x \in [a,b]} |f(x) - p_n(x)| \end{aligned}$$

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Theorem

Let $f \in C[a, b]$, $p \in \mathcal{P}_n$. The following are equivalent:

(1) *p is a minimax polynomial for f of degree n.*

(2) *There exist $x_0 < x_1 < \cdots < x_{n+1}$ contained in $[a, b]$ such that*

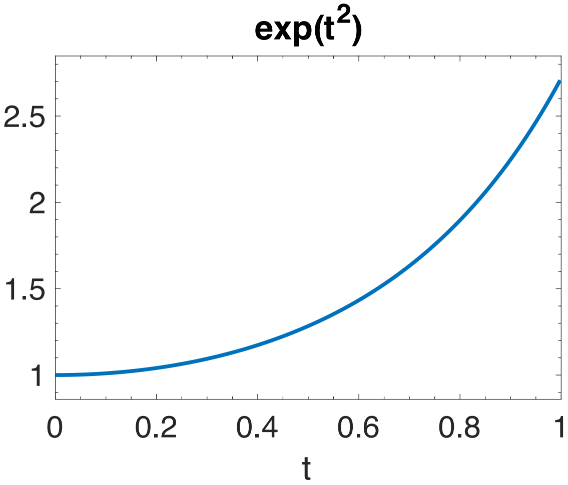
- **(a)** $f(x_j) - p(x_j) = -\{f(x_{j+1}) - p(x_{j+1})\}$ *for* $j = 0, 1, \ldots, n$,
- **(b)** $|f(x_j) - p(x_j)| = \|f - p\|_\infty$ *for* $j = 0, 1, \ldots, n + 1$.

An Example

Suppose f has monotonically increasing derivative on $[a, b]$

e.g. $f(t) = e^{t^2}$ on $[0, 1]$

Find the minimax polynomial p_* of degree 1



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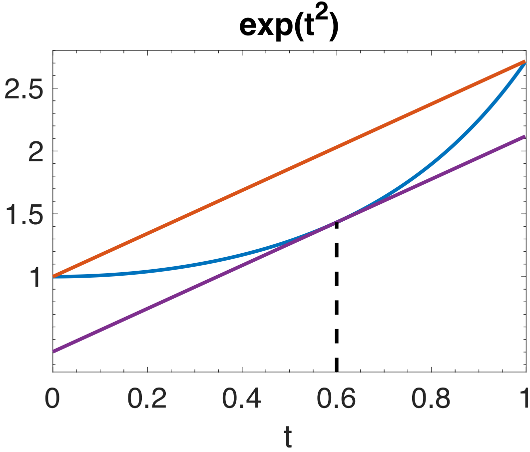

The solution should depend on the secant and tangent lines

$$s(t) = \frac{f(b) - f(a)}{b - a}(t - a) + f(a), \quad \ell(t) = \frac{f(b) - f(a)}{b - a}(t - c) + f(c)$$

where $c \in (a, b)$ is such that $f'(c) = (f(b) - f(a))/(b - a)$.

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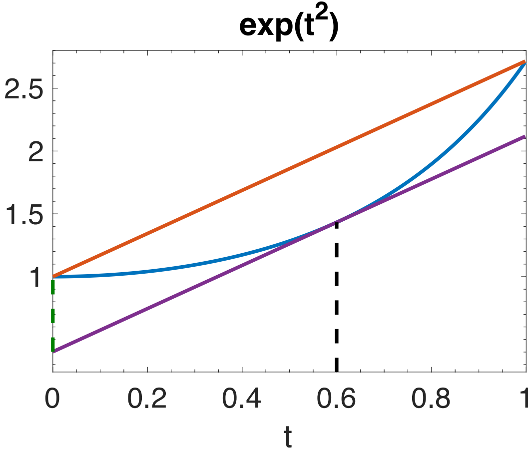

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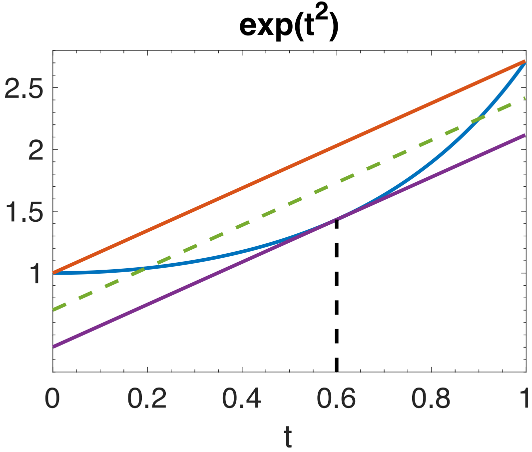

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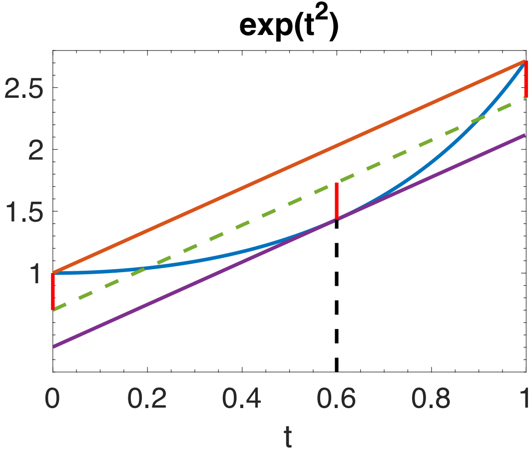

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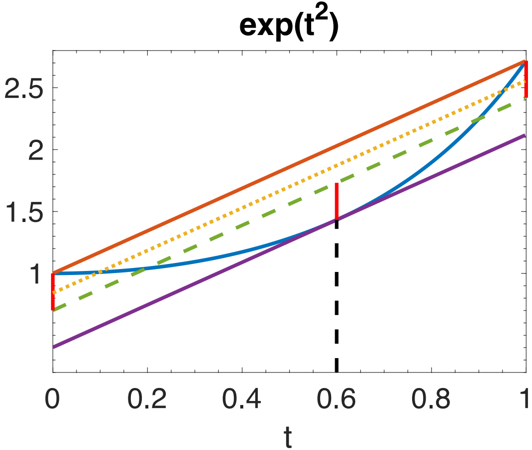

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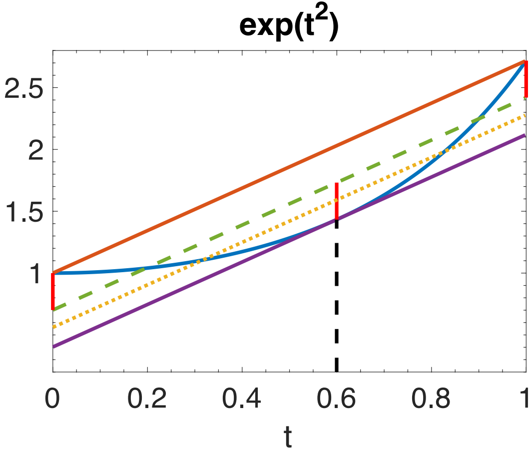

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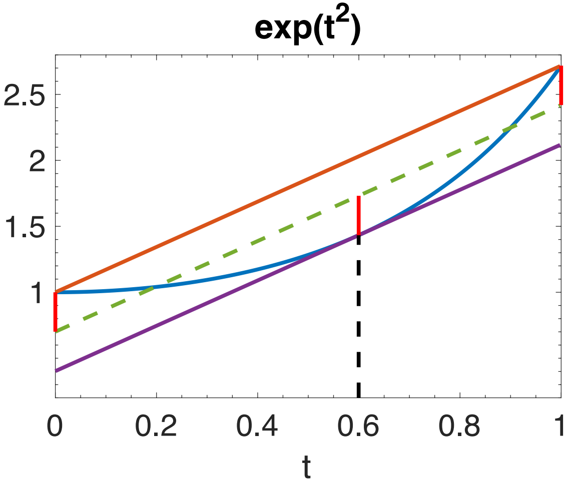

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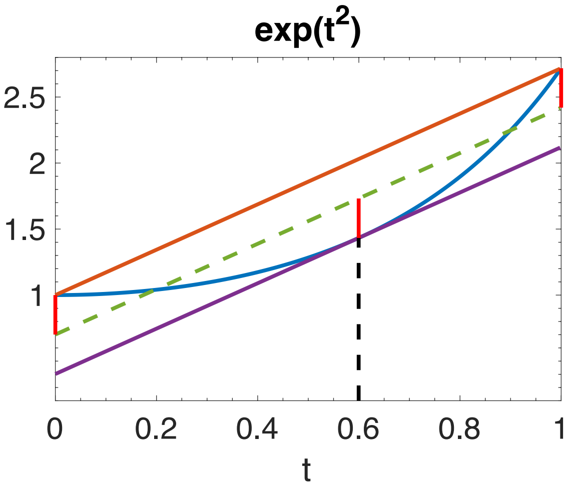

Letting p be the green line, $A := \|f - p\|_\infty$, and
Setting $x_0 = a$, $x_1 = c$, $x_2 = b$, we have

$$f(x_0) - p(x_0) = A, \quad f(x_1) - p(x_1) = -A, \quad f(x_2) - p(x_2) = A$$