# Chebyshev Polynomials

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$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x$$
$$= 4x^3 - 3x$$

# Some Properties of Chebyshev Polynomials

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$$n = 0, 1, 2, 3, \ldots$$

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## Lemma

**(1)** *The leading term of $T_n(x)$ is $2^{n-1}x^n$ for $n \geq 1$.*

**(2)** $|T_n(x)| \leq 1$ *for all* $x \in [-1, 1]$.

**(3)** $|T_n(x_j)| = 1$ *for* $x_j = \cos(j\pi/n)$ $j = 0, \ldots, n$, *furthermore* $T_n(x_j) = -T_n(x_{j+1})$ *for* $j = 0, \ldots, n-1$.

**(4)** $T_n(x_j) = 0$ *for* $x_j = \cos((2j+1)\pi/(2n))$ $j = 0, \ldots, n-1$.

# Minimax Polynomial of Degree $n$ for $x^{n+1}$

**Theorem**

*Letting* $p_*(x) := x^{n+1} - 2^{-n} T_{n+1}(x) \in \mathcal{P}_n$*, we have*

$$\begin{aligned} \|x^{n+1} - p_*\|_\infty &= \min_{p_n \in \mathcal{P}_n} \|x^{n+1} - p_n\|_\infty \\ &= \min_{p_n \in \mathcal{P}_n} \max_{x \in [-1,1]} |x^{n+1} - p_n(x)| \end{aligned}$$

# Minimax Polynomial of Degree $n$ for $x^{n+1}$

Setting $\mathcal{P}_{n+1}^1 := \{c_0 + c_1 x + \cdots + c_n x^n + x^{n+1} \mid c_0, \ldots, c_n \in \mathbb{R}\}$,

$$\min_{\Pi_{n+1} \in \mathcal{P}_{n+1}^1} \|\Pi_{n+1}\|_\infty = \min_{p_n \in \mathcal{P}_n} \|x^{n+1} - p_n\|_\infty$$

# Minimax Polynomial of Degree $n$ for $x^{n+1}$

Setting $\mathcal{P}^1_{n+1} := \{c_0 + c_1 x + \cdots + c_n x^n + x^{n+1} \mid c_0, \ldots, c_n \in \mathbb{R}\}$,

$$\begin{aligned}\min_{\Pi_{n+1} \in \mathcal{P}^1_{n+1}} \|\Pi_{n+1}\|_\infty &= \min_{p_n \in \mathcal{P}_n} \|x^{n+1} - p_n\|_\infty \\ &= \|x^{n+1} - (x^{n+1} - 2^{-n} T_{n+1}(x))\|_\infty\end{aligned}$$

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- $\|\Pi_{n+1}\|_\infty$ over all $\Pi_{n+1} \in \mathcal{P}^1_{n+1}$ is minimized by $2^{-n} T_{n+1}(x)$.
- Optimal roots are the roots of $T_{n+1}(x)$.

# Runge's Example Revisited

$$f(x) = \frac{1}{1+x^2} \qquad x \in [-1, 1]$$

$p_n(x)$ - Lagrange interpolation polynomial for $f(x)$ with interpolation points $x_0, \ldots, x_n$ chosen as the roots of $T_{n+1}(x)$.

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**Interpolation Error**

$$|f(x) - p_n(x)| = \left| \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} \pi_{n+1}(x) \right|$$

where

$\pi_{n+1}(x) = (x - x_0)(x - x_1) \ldots (x - x_n) = 2^{-n} T_{n+1}(x)$.

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It can be verified that $f^{(n+1)}(x) \leq 2^{n+2}$ for all $x \in [-1, 1]$, and $\|\pi_{n+1}\|_\infty = 2^{-n}$, so

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- $\|f - p_n\|_\infty \to 0$ as $n \to \infty$