Orthogonal Polynomials

Inner Product Spaces

Definition

A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ over a vector space V is called an inner product if

(1) (Positivity) $\langle f, f \rangle > 0 \quad \forall f \in V, \ f \not\equiv 0$

(2) (Symmetry) $\langle f, g \rangle = \langle g, f \rangle \quad \forall f, g \in V$

(3) $\langle f, \alpha g \rangle = \alpha \langle f, g \rangle \quad \forall f, g \in V, \ \forall \alpha \in \mathbb{R}$

(4) $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle \quad \forall f, g, h \in V$.

A vector space with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.

Inner Product Spaces

Example

$L^2_w(a,b)$ with

$$\langle f,g\rangle \;:=\; \int_a^b w(x)f(x)g(x)dx$$

is an inner product space.

Induced Norm

Definition

The norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$ is defined as

$$\|f\| := \sqrt{\langle f, f\rangle}.$$

Induced Norm

Definition

The norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$ is defined as

$$\|f\| \quad := \quad \sqrt{\langle f, f\rangle}.$$

Example

$$\|f\|_2 = \sqrt{\int_a^b w(x)f(x)^2 dx}$$

is the norm induced by

$$\langle g, h\rangle = \int_a^b w(x)g(x)h(x)dx.$$

Orthogonality

- Two vectors f, g in an inner product space are called orthogonal if $\langle f, g \rangle = 0$.

Orthogonality

- Two vectors f, g in an inner product space are called orthogonal if $\langle f, g \rangle = 0$.

Definition

A system of orthogonal polynomials $\varphi_0, \ldots, \varphi_n$ in $L^2_w(a, b)$ is such that **(i)** φ_j is exactly of degree j, and

$$\textbf{(ii)} \quad \langle \varphi_j, \varphi_k \rangle = 0 \quad \text{for } k, j \in \{0, 1, \ldots, n\} \text{ s.t. } k \neq j.$$

Gram-Schmidt Procedure

Finding an orthogonal system of polynomials for $L^2_w(a, b)$

$$\varphi_0(x) \equiv 1$$
$$\varphi_j(x) = x^j - \beta_0\varphi_0(x) - \beta_1\varphi_1(x) - \cdots - \beta_{j-1}\varphi_{j-1}(x) \quad j \geq 1$$

where $\beta_k = \langle \varphi_k, x^j \rangle / \langle \varphi_k, \varphi_k \rangle$ $k = 0, 1, \ldots, j-1$.

Chebyshev Polynomials Revisited

$$T_n(x) = \cos(n \cdot \arccos(x))$$

- Orthogonal on $L^2_w(-1, 1)$ with $w(x) = 1/\sqrt{1 - x^2}$

$$\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \cos(n \cdot \arccos(x)) \cos(m \cdot \arccos(x)) dx$$

Chebyshev Polynomials Revisited

Letting θ s.t. $x = \cos(\theta)$

$$\langle T_n, T_m \rangle = \int_{\pi}^{0} \frac{1}{\sqrt{1 - \cos^2(\theta)}} \cos(n\theta)\cos(m\theta)(-\sin(\theta))d\theta$$

Chebyshev Polynomials Revisited

Letting θ s.t. $x = \cos(\theta)$

$$\langle T_n, T_m \rangle = \int_{\pi}^{0} \frac{1}{\sqrt{1 - \cos^2(\theta)}} \cos(n\theta)\cos(m\theta)(-\sin(\theta))d\theta$$
$$= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)d\theta$$

Chebyshev Polynomials Revisited

Letting θ s.t. $x = \cos(\theta)$

$$\begin{aligned}
\langle T_n, T_m \rangle &= \int_{\pi}^{0} \frac{1}{\sqrt{1-\cos^2(\theta)}} \cos(n\theta)\cos(m\theta)(-\sin(\theta))d\theta \\
&= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)d\theta \\
&= \int_{0}^{\pi} \frac{\cos((n+m)\theta)+\cos((n-m)\theta)}{2} d\theta
\end{aligned}$$

Chebyshev Polynomials Revisited

Letting θ s.t. $x = \cos(\theta)$

$$\begin{aligned}
\langle T_n, T_m \rangle &= \int_{\pi}^{0} \frac{1}{\sqrt{1 - \cos^2(\theta)}} \cos(n\theta)\cos(m\theta)(-\sin(\theta))d\theta \\
&= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)d\theta \\
&= \int_{0}^{\pi} \frac{\cos((n+m)\theta) + \cos((n-m)\theta)}{2} d\theta \\
&= \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}
\end{aligned}$$