Roots of Orthogonal Polynomials

Orthogonal Polynomials

Definition

A system of orthogonal polynomials $\varphi_0, \dots, \varphi_n$ in $L^2_w(a, b)$ is such that **(i)** φ_j is exactly of degree j, and

$$\textbf{(ii)} \quad \langle \varphi_j, \varphi_k \rangle = 0 \quad \text{for } k, j \in \{0, 1, \dots, n\} \text{ s.t. } k \neq j.$$

- $\varphi_0(x) = 1$, $\varphi_1(x) = x - 1/2$, $\varphi_2(x) = x^2 - x + 1/6$ in $L^2_w(0, 1)$ with $w(x) \equiv 1$

- $\varphi_n(x) = \cos(n \cdot \arccos(x))$ $n = 0, 1, 2, \dots$ in $L^2_w(-1, 1)$ with $w(x) = 1/\sqrt{1 - x^2}$

Roots

Theorem

Let $\varphi_0, \varphi_1, \ldots, \varphi_n$ be a system of orthogonal polynomials in $L^2_w(a, b)$. Then all roots of ϕ_j, $j \geq 1$ are simple, and lie in (a, b).

Roots

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Polynomial	Space	Roots
$\varphi_1(x) = x - \frac{1}{2}$	$L^2_w(0, 1)$, $w(x) \equiv 1$	$\frac{1}{2}$
$\varphi_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$	$L^2_w(-1, 1)$, $w(x) \equiv 1$	$\pm\frac{1}{\sqrt{3}}$
$\varphi_3(x) = 4x^3 - 3x$	$L^2_w(-1, 1)$, $w(x) = \frac{1}{\sqrt{1-x^2}}$	$0, \pm\frac{\sqrt{3}}{2}$

Proof

Suppose $\eta_1, \ldots, \eta_k$ are the (distinct) points in (a, b) where φ_j changes sign.

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If $k = 0$ (that is φ_j does not change sign on (a, b)),

$$0 \;=\; \langle \varphi_j, 1 \rangle \;=\; \int_a^b w(x)\varphi_j(x)dx$$

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yields a contradiction, as the last integral cannot be zero (i.e., $w(x)\varphi_j(x) > 0\ \forall x \in (a, b)$ or $w(x)\varphi_j(x) < 0\ \forall x \in (a, b)$ excluding possibly x that are the roots of φ_j).

Proof

Suppose $k < j$. Letting $\phi(x) := \prod_{\ell=1}^{k}(x - \eta_\ell)$, we have

$$0 \;=\; \langle \varphi_j, \phi \rangle \;=\; \int_a^b w(x)\varphi_j(x)\phi(x)dx.$$

Proof

Suppose $k < j$. Letting $\phi(x) := \prod_{\ell=1}^{k}(x - \eta_\ell)$, we have

$$0 = \langle \varphi_j, \phi \rangle = \int_a^b w(x)\varphi_j(x)\phi(x)dx.$$

But again the last integral cannot be zero, so we end up with a contradiction.
($w(x)\varphi_j(x)\phi(x) > 0 \forall x \in (a, b)$ or $w(x)\varphi_j(x)\phi(x) < 0 \forall x \in (a, b)$ excluding possibly x that are the roots of φ_j, ϕ).

Proof

- Consequently, $k = j$, that is φ_j must change sign at j distinct points $\eta_1, \ldots, \eta_j \in (a, b)$.

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- $\eta_1, \ldots, \eta_j \in (a, b)$ must consist of all roots of φ_j, all simple.