Initial Value Problems for ODEs

Problem

Initial Value Problem

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \quad \text{(initial condition)} \end{aligned}$$

$f : \mathbb{R}^2 \to \mathbb{R}, \quad y(x) : [x_0, X_M] \to \mathbb{R}$

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- Assumption: solution $y(x) : [x_0, X_M] \to \mathbb{R}$ is unique.

General Numerical Approach

Letting $x_j = x_0 + jh$ for $j = 0, 1, \ldots, N$, where $h := (X_M - x_0)/N$

find $\quad y_j \approx y(x_j) \quad j = 0, 1, \ldots, N$

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Euler's Method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$n = 0, 1, \ldots, N-1$

One-Step Methods

$$y_{n+1} = y_n + h \cdot \phi(x_n, y_n; h) \qquad n = 0, 1, \ldots, N-1$$

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Truncation Error

$$T_n := \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h) \qquad n = 0, 1, \ldots, N-1$$

One-Step Methods

Theorem

Suppose there exists a $\widetilde{\gamma} > 0$ *s.t.*

$$|\phi(x, u; h) - \phi(x, v; h)| \leq \widetilde{\gamma}|u - v| \quad \forall (x, u), (x, v) \in \mathbb{R}^2,$$

and for all h *small enough, say* $h \leq h_0$*.*

Then provided $h \leq h_0$*, we have*

$$|e_n| \quad \leq \quad \frac{T}{\widetilde{\gamma}}(e^{\widetilde{\gamma}(x_n - x_0)} - 1) \quad n = 0, 1, \ldots, N$$

where $T := \max\{|T_j| \mid j = 0, 1, \ldots, N-1\}$*.*

Proof

We have

$$\begin{aligned} y_{n+1} &= y_n + h\phi(x_n, y_n; h), \\ y(x_{n+1}) &= y(x_n) + h\phi(x_n, y(x_n); h) + hT_n. \end{aligned}$$

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Subtracting the first eqn. from the second yields

$$e_{n+1} = e_n + h\{\phi(x_n, y(x_n); h) - \phi(x_n, y_n; h)\} + hT_n.$$

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Considering $h \leq h_0$ and exploiting Lipschitz continuity of ϕ, we deduce

$$|e_{n+1}| \leq |e_n| + h\widetilde{\gamma}|y(x_n) - y_n| + hT$$

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$$\begin{aligned} |e_{n+1}| &\leq |e_n| + h\widetilde{\gamma}|y(x_n) - y_n| + hT \\ &= (1 + h\widetilde{\gamma})|e_n| + hT. \end{aligned}$$

Proof

Now combining $|e_{n+1}| \leq (1 + h\widetilde{\gamma})|e_n| + hT$ and $e_0 = 0$, by induction, it can be shown that

$$|e_n| \leq \frac{T}{\widetilde{\gamma}}\{(1 + h\widetilde{\gamma})^n - 1\} \quad n = 0, 1, \ldots, N.$$

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$$\begin{aligned} |e_n| &\leq \frac{T}{\widetilde{\gamma}}\{e^{h\widetilde{\gamma}n} - 1\} \\ &= \frac{T}{\widetilde{\gamma}}\{e^{\widetilde{\gamma}(x_n - x_0)} - 1\} \end{aligned}$$

for $n = 0, 1, \ldots, N$.

Truncation Error of Euler's Method

Euler's method

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$$T \leq \frac{M_2}{2} h, \quad M_2 := \max_{x \in [x_0, X_M]} |y''(x)|$$

Global Error of Euler's Method

$$
\begin{aligned}
|e_n| &\leq \frac{M_2}{2}\left\{\frac{e^{\gamma(x_n - x_0)} - 1}{\gamma}\right\} h \\
&\leq \frac{M_2}{2}\left\{\frac{e^{\gamma(X_m - x_0)} - 1}{\gamma}\right\} h
\end{aligned}
$$

γ - Lipschitz constant for $f(x, y)$ w.r.t. y