Taylor's Theorem for Functions of Two Variables

December 17, 2018

Theorem 0.1. *Let* $f := f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ *be three times differentiable. Then*

$$\begin{aligned} f(a+hp, b+hq) &= f(a,b) + \{f_x(a,b)(hp) + f_y(a,b)(hq)\} \\ &+ \frac{1}{2}\left\{f_{xx}(a,b)(hp)^2 + 2f_{xy}(a,b)(hp)(hq) + f_{yy}(a,b)(hq)^2\right\} \\ &+ O(h^3). \end{aligned}$$

Proof. Let $F(t) := f(a + thp, b + thq)$, and apply Taylor's theorem to F to obtain

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2} + \frac{F'''(\varepsilon)}{6} \qquad \exists \varepsilon \in (0, 1)$$

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leading us to

$$\underbrace{f(a+hp, b+hq)}_{F(1)} = \underbrace{f(a,b)}_{F(0)} + \underbrace{\{f_x(a,b)(hp) + f_y(a,b)(hq)\}}_{F'(0)}$$
$$+ \underbrace{\left\{f_{xx}(a,b)(hp)^2 + 2f_{xy}(a,b)(hp)(hq) + f_{yy}(a,b)(hq)^2\right\}}_{F''(0)}/2$$
$$+ \underbrace{O(h^3)}_{F'''(\varepsilon)/6}.$$