Summary of Lecture 1

September 19, 2018

$$f(x) = 0 \iff \underbrace{f(x) + x}_{g(x)} = x$$

- $f, g : \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$

Existence of a fixed point on $[a, b]$
(*) If g maps $[a, b]$ into $[a, b]$

Theorem 0.1 (Brouwer's Fixed Point Thm)**.** *Suppose* $g(x) \in [a, b]$ *for all* $x \in [a, b]$. *Then*

$$g(x_*) = x_* \quad \exists x_* \in [a, b].$$

Uniqueness of a fixed point on $[a, b]$
In addition to (*), if g is a contraction on $[a, b]$.

Computation
The simple iteration

$$\{x_k\},\ x_{k+1} = g(x_k)$$

converges to this unique fixed point.

Theorem 0.2 (Contraction Mapping)**.** *Suppose* $g(x) \in [a, b]$ *for all* $x \in [a, b]$*. Additionally, suppose there exists an* $L < 1$ *such that*

$$|g(x) - g(y)| \ \leq \ L|x - y| \qquad \forall x, y \in [a, b].$$

(1) $g(x_*) = x_*$ *for a unique* $x_* \in [a, b]$.

(2) $\{x_k\},\ x_{k+1} = g(x_k)$ *is s.t.* $\lim_{k\to\infty} x_k = x_*$ *for all* $x_0 \in [a, b]$.

Example.
$f(x) = x^2 - 2$ on $[1, 2]$.

(1) $g(x) = x^2 - 2 + x$, $\qquad x_{k+1} = x_k^2 - 2 + x_k$

(2) $h(x) = \frac{2-x^2}{4} + x$, $\qquad x_{k+1} = \frac{2-x_k^2}{4} + x_k$

(3) $k(x) = \frac{x^2+2}{2x}$, $\qquad x_{k+1} = \frac{x_k^2+2}{2x_k}$

Taylor's theorem with 2nd order remainder

Let $g : [a, b] \to \mathbb{R}$ be such that g, g' are continuous on $[a, b]$ and g'' exists on (a, b). Then

$$g(b) = g(a) + g'(a)(b-a) + \frac{g''(\varepsilon)}{2}(b-a)^2 \quad \exists \varepsilon \in (a, b).$$

Theorem 0.3 (Taylor's theorem with pth order remainder)**.** *Let $g : [a, b] \to \mathbb{R}$ be such that $g, g', \ldots, g^{(p-1)}$ are continuous on $[a, b]$ and $g^{(p)}$ exists on (a, b). Then*

$$g(b) = g(a) + \sum_{j=1}^{p-1} \frac{g^{(j)}(a)}{j!}(b-a)^j + \frac{g^{(p)}(\varepsilon)}{p!}(b-a)^p \quad \exists \varepsilon \in (a, b).$$