Convergence of the Secant Method and the Bisection Method

September 26, 2018

Sequence $\{x_k\}$ by the secant method

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}, \quad k \geq 2$$

given $x_0, x_1 \in \mathbb{R}$.

Theorem 0.1. *Let x_* be s.t. $f(x_*) = 0$, and $f'(x)$ is continuous in a neighborhood of x_*. Furthermore, suppose $f'(x_*) \neq 0$.*

(i) *The sequence $\{x_k\}$ by the secant method converges to x_* for all $x_0, x_1 \in \mathbb{R}, x_0 \neq x_1$ sufficiently close to x_*.*

(ii) *The order of this convergence is superlinear, provided $f'(x)$ is Lipschitz continuous in a neighborhood of x_*.*

Proof of (ii)

By the fundamental theorem

$$0 \;=\; f(x_*) \;=\; f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx$$

Proof of (ii)

By the fundamental theorem

$$\begin{aligned} 0 \;=\; f(x_*) &= f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx \\ &= f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt \end{aligned}$$

Proof of (ii)

By the fundamental theorem

$$
\begin{aligned}
0 = f(x_*) &= f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx \\
&= f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt
\end{aligned}
$$

Divide both sides by $f'(x_k)$ to have

$$
0 = \frac{f(x_k)}{f'(x_k)} + \frac{1}{f'(x_k)} \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt,
$$

Proof of (ii)

By the fundamental theorem

$$\begin{aligned} 0 = f(x_*) &= f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx \\ &= f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt \end{aligned}$$

Divide both sides by $f'(x_k)$ to have

$$0 = \frac{f(x_k)}{f'(x_k)} + \frac{1}{f'(x_k)} \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt,$$

then letting $h(x_k, x_{k-1}) := \{f(x_k) - f(x_{k-1})\} / \{x_k - x_{k-1}\}$

$$\begin{aligned} 0 = & \frac{f(x_k)}{h(x_k, x_{k-1})} + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})}\right\} + (x_* - x_k) \\ & + \frac{1}{f'(x_k)} \int_0^1 \{f'(x_k + t(x_* - x_k)) - f'(x_k)\}\,(x_* - x_k)\,dt \end{aligned}$$

Proof of (ii)

By the fundamental theorem

$$\begin{aligned} 0 \;=\; f(x_*) \;&=\; f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx \\ &=\; f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt \end{aligned}$$

Divide both sides by $f'(x_k)$ to have

$$0 = \frac{f(x_k)}{f'(x_k)} + \frac{1}{f'(x_k)} \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt,$$

then letting $h(x_k, x_{k-1}) := [f(x_k) - f(x_{k-1})] \,/\, [x_k - x_{k-1}]$

$$\begin{aligned} 0 \;=\; & \underbrace{\frac{f(x_k)}{h(x_k, x_{k-1})}}_{x_k - x_{k+1}} + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})}\right\} + (x_* - x_k) \\ & + \frac{1}{f'(x_k)} \int_0^1 \left\{f'(x_k + t(x_* - x_k)) - f'(x_k)\right\}(x_* - x_k)\,dt. \end{aligned}$$

Proof of (ii)

By the fundamental theorem

$$\begin{aligned} 0 \;=\; f(x_*) \;&=\; f(x_k) + \int_{x_k}^{x_*} f'(x)\,dx \\ &=\; f(x_k) + \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt \end{aligned}$$

Divide both sides by $f'(x_k)$ to have

$$0 = \frac{f(x_k)}{f'(x_k)} + \frac{1}{f'(x_k)} \int_0^1 f'(x_k + t(x_* - x_k))(x_* - x_k)\,dt,$$

then letting $h(x_k, x_{k-1}) := [f(x_k) - f(x_{k-1})] \,/\, [x_k - x_{k-1}]$

$$\begin{aligned} 0 \;=\; & \underbrace{\frac{f(x_k)}{h(x_k, x_{k-1})}}_{x_k - x_{k+1}} + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})}\right\} + (x_* - x_k) \\ & + \frac{1}{f'(x_k)} \int_0^1 \{f'(x_k + t(x_* - x_k)) - f'(x_k)\}\,(x_* - x_k)\,dt. \end{aligned}$$

Hence,

$$\begin{aligned} 0 \;=\; & (x_* - x_{k+1}) + f(x_k)\left\{\frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})}\right\} \\ & + \frac{1}{f'(x_k)} \int_0^1 \{f'(x_k + t(x_* - x_k)) - f'(x_k)\}\,(x_* - x_k)\,dt. \end{aligned}$$

Now let γ be s.t.

$$|f'(x) - f'(y)| \leq \gamma |x - y|$$

$\forall x, y$ in a neighborhood $\mathcal{I} = [x_* - \delta, x_* + \delta]$ of x_*.

Now let γ be s.t.

$$|f'(x) - f'(y)| \le \gamma |x - y|$$

$\forall x, y$ in a neighborhood $\mathcal{I} = [x_* - \delta, x_* + \delta]$ of x_*.

Consider k large enough so that $x_k \in \mathcal{I}$ for all such large k.

Now let γ be s.t.

$$|f'(x) - f'(y)| \leq \gamma |x - y|$$

$\forall x, y$ in a neighborhood $\mathcal{I} = [x_* - \delta, x_* + \delta]$ of x_*.

Consider k large enough so that $x_k \in \mathcal{I}$ for all such large k.

Using $|a + b| \leq |a| + |b|$ and $|ab| \leq |a||b|$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |x_* - x_{k+1}| &\leq |f(x_k)| \left| \frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})} \right| \\ &+ \left| \frac{1}{f'(x_k)} \right| \int_0^1 |f'(x_k + t(x_* - x_k)) - f'(x_k)| \, |x_* - x_k| \; dt. \end{aligned}$$

Now let γ be s.t.

$$|f'(x) - f'(y)| \le \gamma |x - y|$$

$\forall x, y$ in a neighborhood $\mathcal{I} = [x_* - \delta, x_* + \delta]$ of x_*.

Consider k large enough so that $x_k \in \mathcal{I}$ for all such large k.

Using $|a + b| \le |a| + |b|$ and $|ab| \le |a||b|$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |x_* - x_{k+1}| \ &\le \ |f(x_k)| \left| \frac{1}{f'(x_k)} - \frac{1}{h(x_k, x_{k-1})} \right| \\ &\quad + \left| \frac{1}{f'(x_k)} \right| \int_0^1 |f'(x_k + t(x_* - x_k)) - f'(x_k)| \, |x_* - x_k| \ dt. \end{aligned}$$

Now by the mean value theorem, as well as the Lipschitz continuity of $f'(x)$ in $\mathcal{I}$,

$$\begin{aligned} |x_* - x_{k+1}| \ &\le \ |f'(\eta_k)| \, |x_* - x_k| \, \frac{|h(x_k, x_{k-1}) - f'(x_k)|}{|f'(x_k)| \, |h(x_k, x_{k-1})|} \\ &\quad + \left| \frac{1}{f'(x_k)} \right| \int_0^1 \gamma t \, |x_* - x_k|^2 \ dt. \end{aligned}$$

for some $\eta_k \in \mathcal{I}$.

Finally, divide both sides by $|x_* - x_k|$, take the limit as $k \to \infty$ to obtain

$$\lim_{k\to\infty} \frac{|x_* - x_{k+1}|}{|x_* - x_k|} \leq \lim_{k\to\infty} |f'(\eta_k)| \frac{|h(x_k, x_{k-1}) - f'(x_k)|}{|f'(x_k)|\,|h(x_k, x_{k-1})|} + \lim_{k\to\infty} \left|\frac{1}{f'(x_k)}\right| \frac{\gamma\,|x_* - x_k|}{2}\, dt = 0.$$

Bisection Method.

Suppose $a_0, b_0 \in \mathbb{R}, a_0 < b_0$ are s.t.

$$f(a_0)f(b_0) < 0.$$

Bisection Method.

Suppose $a_0, b_0 \in \mathbb{R}, a_0 < b_0$ are s.t.

$$f(a_0)f(b_0) < 0.$$

Generate sequences $\{a_k\},\ \{b_k\},\ \{c_k\}$ by the rules

$$c_k \quad := (a_k + b_k)/2, \qquad \text{if } f(c_k) = 0\text{, terminate}$$

Bisection Method.

Suppose $a_0, b_0 \in \mathbb{R}, a_0 < b_0$ are s.t.

$$f(a_0)f(b_0) < 0.$$

Generate sequences $\{a_k\},\ \{b_k\},\ \{c_k\}$ by the rules

$$c_k \quad := (a_k + b_k)/2, \qquad \text{if } f(c_k) = 0\text{, terminate}$$

$$a_{k+1} := a_k, \quad b_{k+1} := c_k, \quad \text{if } f(a_k)f(c_k) < 0$$

Bisection Method.

Suppose $a_0, b_0 \in \mathbb{R}, a_0 < b_0$ are s.t.

$$f(a_0)f(b_0) < 0.$$

Generate sequences $\{a_k\},\ \{b_k\},\ \{c_k\}$ by the rules

$$c_k \ := (a_k + b_k)/2, \quad \text{if } f(c_k) = 0\text{, terminate}$$

$$a_{k+1} := a_k,\ \ b_{k+1} := c_k,\ \ \text{if } f(a_k)f(c_k) < 0$$

$$a_{k+1} := c_k,\ \ b_{k+1} := b_k,\ \ \text{if } f(a_k)f(c_k) > 0$$

Convergence

$$|c_k - x_*| \le (b_0 - a_0)2^{-(k+1)} \qquad \exists x_* \in [a_0, b_0] \text{ s.t. } f(x_*) = 0.$$

- $\lim_{k\to\infty} c_k \; = \; x_*$
- converges at least linearly