Reminders from Lecture 7

October 9, 2018

$\|\cdot\| : \mathcal{V} \to \mathbb{R}$ is a norm if

- (Positivity) $\|v\| > 0$ for all nonzero $v \in \mathcal{V}$,
- (Homogeneity) $\|\alpha v\| = |\alpha|\|v\|$ for all $v \in \mathcal{V}$, all $\alpha \in \mathbb{R}$,
- (Triangle Inequality) $\|v+w\| \leq \|v\| + \|w\|$ for all $v, w \in \mathcal{V}$.

Common norms in $\mathbb{R}^n$

$$\text{(2-norm)} \qquad \|v\|_2 := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\text{(1-norm)} \qquad \|v\|_1 := |v_1| + |v_2| + \cdots + |v_n|$$

$$\infty\text{-norm} \qquad \|v\|_\infty := \max_{j=1,\ldots,n} |v_j|$$

For $\| \cdot \|_2$ positivity and homogeneity are straightforward.

Triangle inequality follows from Cauchy-Schwarz inequality.

Theorem 0.1 (Cauchy-Schwarz Inequality)**.** *For every* $v, w \in \mathbb{R}^n$*, we have*

$$\left| \sum_{j=1}^{n} v_j w_j \right| \quad \leq \quad \|v\|_2 \|w\|_2.$$

Proof of the Cauchy-Schwarz inequality.

Consider following nonnegative quadratic polynomial in t.

$$q(t) \quad := \quad \|v + tw\|_2^2 \quad = \quad \sum_{j=1}^{n} v_j^2 + 2t \sum_{j=1}^{n} v_j w_j + t^2 \sum_{j=1}^{n} w_j^2$$

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The discriminant of $q(t)$ must be nonpositive, that is

$$\left(2 \sum_{j=1}^{n} v_j w_j \right)^2 - 4 \left(\sum_{j=1}^{n} v_j^2 \sum_{j=1}^{n} w_j^2 \right) \leq 0$$

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which in turn implies the Cauchy-Schwarz inequality

$$\left|\sum_{j=1}^{n} v_j w_j\right| \quad \leq \quad \|v\|_2 \|w\|_2.$$