Cauchy Sequences in $\mathbb{R}^n$

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(1) A subset D of $\mathbb{R}^n$ is open, if for every $\xi \in D$ there exists an $\varepsilon > 0$ such that

$$B_\varepsilon(\xi) \quad \subseteq \quad D,$$

where $B_\varepsilon(\xi) := \{x \in \mathbb{R}^n \mid \|x - \xi\|_\infty < \varepsilon\}$.

(2) A subset D of $\mathbb{R}^n$ is closed if $\mathbb{R}^n \backslash D$ is open.

(3) An open neighborhood $N(\xi)$ of $\xi \in \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ that contains ξ.

Definition 0.1. *A function* $g : D \to \mathbb{R}^n$ *with* D *denoting a nonempty subset of* $\mathbb{R}^n$ *is continuous at* $\xi \in D$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\|g(x) - g(\xi)\|_\infty \leq \varepsilon \qquad \forall x \in \overline{B}_\delta(\xi) \cap D,$$

where $\overline{B}_\varepsilon(\xi) := \{x \in \mathbb{R}^n \mid \|x - \xi\|_\infty \leq \varepsilon\}$.

The function g *is said to be continuous on* D *if* g *is continuous at every* $\xi \in D$.

Theorem 0.2. *Let*

- *the function* $g : D \to \mathbb{R}^n$ *with* D *denoting a nonempty subset of* $\mathbb{R}^n$ *be continuous on* D*, and*
- *the sequence* $\{x^{(k)}\}$ *with* $x^{(k)} \in D$ *for all* k *be convergent with the limit in* D*.*

Then

$$\lim_{k\to\infty} g(x^{(k)}) \quad = \quad g(\lim_{k\to\infty} x^{(k)}).$$

Definition 0.3. *A sequence in $\mathbb{R}^n$ is said to be Cauchy if for every $\varepsilon > 0$ there exists an integer $K > 0$ such that*

$$\|x^{(m)} - x^{(k)}\|_\infty \leq \varepsilon \qquad \forall m, k \geq K.$$

Example.

The sequence $\{s^{(k)}\}$ in $\mathbb{R}$ with

$$s^{(k)} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^k$$

is Cauchy.

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The sequence $\{s^{(k)}\}$ in $\mathbb{R}$ with

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is Cauchy.

Indeed, for every m, k with $m > k$ we have

$$|s^{(m)} - s^{(k)}| \leq \left(\frac{1}{2}\right)^{k+1} + \cdots + \left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^{k+1} \frac{1 - \left(\frac{1}{2}\right)^{m-k}}{1 - 1/2} \leq \left(\frac{1}{2}\right)^k.$$

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For every $\varepsilon > 0$, we can choose $K \in \mathbb{Z}^+$ so that $(1/2)^K \leq \varepsilon$.
Then for every $m, k \geq K$ we have

$$|s^{(m)} - s^{(k)}| \leq \left(\frac{1}{2}\right)^K \leq \varepsilon.$$

Theorem 0.4. *The following are equivalent:*

(i) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is Cauchy.*

(ii) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is convergent.*

Theorem 0.5. *The following are equivalent:*

(i) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is Cauchy.*

(ii) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is convergent.*

Proof of (ii) $\Longrightarrow$ (i)
Suppose $\lim_{k\to\infty} x^{(k)} = \xi$.
Then for every $\varepsilon > 0$ there exists a K such that

$$\|x^{(k)} - \xi\|_\infty \quad \leq \quad \varepsilon/2 \qquad \forall k \geq K.$$

Theorem 0.6. *The following are equivalent:*

(i) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is Cauchy.*

(ii) *The sequence* $\{x^{(k)}\}$ *in* $\mathbb{R}^n$ *is convergent.*

Proof of (ii) $\Longrightarrow$ **(i)**
Suppose $\lim_{k\to\infty} x^{(k)} = \xi$.
Then for every $\varepsilon > 0$ there exists a K such that

$$\|x^{(k)} - \xi\|_\infty \leq \varepsilon/2 \qquad \forall k \geq K.$$

But then for every $m, k \geq K$, we have

$$\|x^{(m)}-x^{(k)}\|_\infty = \|(x^{(m)}-\xi)+(\xi-x^{(k)})\|_\infty \leq \|x^{(m)}-\xi\|_\infty+\|x^{(k)}-\xi\|_\infty \leq \varepsilon,$$

and the proof is complete.