MATH 305: Numerical Anaysis

Instructor: Emre Mengi

Fall Semester 2018
Final Examination

NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

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#2	20	
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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 135 minutes.

Problem 1. Given points

$$(x_0, y_0) = (1, 2), \quad (x_1, y_1) = (2, 1), \quad (x_2, y_2) = (4, -1)$$

in $\mathbb{R}^2$, consider the problem of finding a line $\ell(x) = a_1 x + a_0$ that best fits these points, that is the problem of determining $a_0, a_1 \in \mathbb{R}$ such that

$$\sqrt{\sum_{j=0}^{2} [\ell(x_j) - y_j]^2} \tag{1}$$

is as small as possible.

(a) (5 points) Write down a least squares problem in the form

$$\min_{a_0, a_1 \in \mathbb{R}} \left\| A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} - b \right\|_2$$

for which the minimizing a_0, a_1 also minimizes (1).

(b) (10 points) Solve the least squares problem in part (a) by exploiting the associated normal equation, hence determine the line $\ell(x)$ minimizing (1).

Problem 2.

(a) **(10 points)** Write down the Lagrange interpolation polynomial $p_2(x)$ of degree 2 for $f(x) = \cos((\pi x)/2)$ with interpolation points $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$.

(b) **(10 points)** Now let $p_n(x)$ denote the Lagrange interpolation polynomial of degree n for $f(x) = \cos((\pi x)/2)$ with interpolation points $x_j = jh$ for $j = 0, 1, 2, \ldots, n$ where $h := 1/n$.

Show that for every $x \in [0, 1]$ the following holds:

$$\lim_{n\to\infty} [\cos((\pi x)/2) - p_n(x)] \;=\; 0.$$

Problem 3. (15 points) Suppose that $\mathcal{P}_n$ (the vector space of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree at most n) is equipped with the inner product

$$\langle p, q \rangle := \int_a^b p(x)q(x)dx$$

on a prescribed interval $[a, b] \subset \mathbb{R}$.

It follows from the Gram-Schmidt procedure that there exists a monic polynomial $\varphi_n \in \mathcal{P}_n$ such that

$$\langle \varphi_n, r \rangle \;\; = \;\; 0 \qquad \forall r \in \mathcal{P}_{n-1}. \tag{2}$$

Show that the monic polynomial φ_n satisfying (2) is unique.

(Note: Recall that a monic polynomial of degree n has the form

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x_0 + a_0,$$

that is the coefficient of the leading term is one.)

Problem 4. (15 points) Consider the integral

$$I(f) := \int_{-2}^{2} \frac{1}{\sqrt{1 - x^2/4}} f(x)dx.$$

Derive a quadrature formula in the form

$$Q(f) := w_0 f(x_0) + w_1 f(x_1)$$

for this integral such that $Q(f) = I(f)$ for all $f \in \mathcal{P}_3$.

Problem 5. Let us consider the initial value problem

$$\begin{aligned} y' &= f(x,y), \\ y(x_0) &= y_0 \end{aligned} \tag{3}$$

where $y : [x_0, X_M] \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that this initial value problem has a unique solution y that is differentiable infinitely many times.

A one-step method can be defined for the solution of (3) and for a given positive integer N based on the update rule

$$y_{n+1} = y_n + hf\left(x_n + \frac{3h}{4}\ ,\ y_n + \frac{3h}{4} f(x_n, y_n)\right) \quad n = 0, 1, 2, \ldots, N-1, \tag{4}$$

where $x_n := x_0 + nh$ and $h := (X_m - x_0)/N$.

(a) **(5 points)** Is the one-step method based on the update rule (4) consistent?

(b) **(15 points)** Determine the order of accuracy for this one-step method defined by (4).

Problem 6. (15 points) Consider the ∞-norm

$$\|f\|_\infty := \max_{x\in[-1,1]} |f(x)| \tag{5}$$

defined over $C[-1,1]$, the space of continuous functions on $[-1,1]$.

Find the polynomial $p_* \in \mathcal{P}_2$ such that

$$\|(2x^3 + x^2) - p_*\|_\infty \quad = \quad \min_{p\in\mathcal{P}_2} \|(2x^3 + x^2) - p\|_\infty$$

where $\|\cdot\|_\infty$ is defined as in (5).