Solutions

MATH 305: Numerical Anaysis

Instructor: Emre Mengi

Fall Semester 2018
Final Examination

NAME ________________

STUDENT ID ________________

SIGNATURE ________________

#1	15	
#2	20	
#3	15	
#4	15	
#5	20	
#6	15	
Σ	100	

- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 135 minutes.

Problem 1. Given points

$$(x_0, y_0) = (1, 2), \quad (x_1, y_1) = (2, 1), \quad (x_2, y_2) = (4, -1)$$

in $\mathbb{R}^2$, consider the problem of finding a line $\ell(x) = a_1 x + a_0$ that best fits these points, that is the problem of determining $a_0, a_1 \in \mathbb{R}$ such that

$$\sqrt{\sum_{j=0}^{2} [\ell(x_j) - y_j]^2} \tag{1}$$

is as small as possible.

(a) (5 points) Write down a least squares problem in the form

$$\min_{a_0, a_1 \in \mathbb{R}} \left\| A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} - b \right\|_2$$

for which the minimizing a_0, a_1 also minimizes (1).

(b) (10 points) Solve the least squares problem in part (a) by exploiting the associated normal equation, hence determine the line $\ell(x)$ minimizing (1).

(a) Would like to solve

$$\min_{a_0, a_1 \in \mathbb{R}} \sqrt{\sum_{j=0}^{2} [\ell(x_j) - y_j]^2}$$

$$= \min_{a_0, a_1 \in \mathbb{R}} \left\| \begin{bmatrix} \ell(x_0) - y_0 \\ \ell(x_1) - y_1 \\ \ell(x_2) - y_2 \end{bmatrix} \right\|_2 = \min_{a_0, a_1 \in \mathbb{R}} \left\| \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} - \underbrace{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}_{b} \right\|_2$$

(b) Associated normal eqn.

$$A^T A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} a_0 = -3a_1 \\ 3a_0 + 7a_1 = 2 \end{array}$$

$$\Rightarrow a_1 = -1 \quad a_0 = 3$$

$$\boxed{\begin{array}{l} \ell(x) = -x + 3 \\ \text{minimizes } (1) \end{array}}$$

Problem 2.

(a) **(10 points)** Write down the Lagrange interpolation polynomial $p_2(x)$ of degree 2 for $f(x) = \cos((\pi x)/2)$ with interpolation points $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$.

(b) **(10 points)** Now let $p_n(x)$ denote the Lagrange interpolation polynomial of degree n for $f(x) = \cos((\pi x)/2)$ with interpolation points $x_j = jh$ for $j = 0, 1, 2, \ldots, n$ where $h := 1/n$.

Show that for every $x \in [0, 1]$ the following holds:

$$\lim_{n\to\infty} [\cos((\pi x)/2) - p_n(x)] = 0.$$

(a) $$p_2(x) = \frac{(x-1/2)(x-1)}{(0-1/2)(0-1)} f(0) + \frac{(x-0)(x-1)}{(1/2-0)(1/2-1)} f(1/2) + \frac{(x-0)(x-1/2)}{(1-0)(1-1/2)} f(1)$$

$$= 2(x^2 - 3/2\, x + 1/2) - 2\sqrt{2}(x^2 - x)$$

$$= (2-2\sqrt{2})x^2 + (2\sqrt{2}-3)x + 1$$

(b) By the Lagrange interpolation polynomial error result, for every $x \in [0,1]$

$$\cos((\pi x)/2) - p_n(x) = \left.\frac{d^{n+1}\cos((\pi x)/2)}{dx^{n+1}}\right|_{x=\xi} \cdot \prod_{j=0}^{n}(x-x_j) \Big/ (n+1)!$$

$\exists\, \xi \in (0,1)$.

Now notice that $\left|\prod_{j=0}^{n}(x-x_j)\right| \leq 1 \quad \forall x \in [0,1]$, additionally $\left|\frac{d^{n+1}\cos((\pi x)/2)}{dx^{n+1}}\right| \leq \left(\frac{\pi}{2}\right)^{n+1} \quad \forall x \in (0,1)$.

Hence,

$$|\cos((\pi x)/2) - p_n(x)| \leq \left(\frac{\pi}{2}\right)^{n+1} \Big/ (n+1)!$$

As $\lim_{n\to\infty} (\pi/2)^{n+1}/(n+1)! = 0$, we have

$$\lim_{n\to\infty} |\cos((\pi x)/2) - p_n(x)| = 0.$$

Problem 3. (15 points) Suppose that $\mathcal{P}_n$ (the vector space of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree at most n) is equipped with the inner product

$$\langle p, q \rangle := \int_a^b p(x)q(x)dx$$

on a prescribed interval $[a, b] \subset \mathbb{R}$.

It follows from the Gram-Schmidt procedure that there exists a monic polynomial $\varphi_n \in \mathcal{P}_n$ such that

$$\langle \varphi_n, r \rangle \; = \; 0 \qquad \forall r \in \mathcal{P}_{n-1}. \tag{2}$$

Show that the monic polynomial φ_n satisfying (2) is unique.

(Note: Recall that a monic polynomial of degree n has the form

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x_0 + a_0,$$

that is the coefficient of the leading term is one.)

Let us suppose $\varphi_n, \phi_n \in \mathcal{P}_n$ are monic polynomials s.t.

(+) $\langle \varphi_n, r \rangle = \langle \phi_n, r \rangle = 0 \quad \forall r \in \mathcal{P}_{n-1}$.

But then $\varphi_n - \phi_n \in \mathcal{P}_{n-1}$, so by (+)

$$\langle \varphi_n, \varphi_n - \phi_n \rangle = \langle \phi_n, \varphi_n - \phi_n \rangle = 0$$

$$\Longrightarrow \langle \varphi_n - \phi_n, \varphi_n - \phi_n \rangle = 0$$

$$\Longrightarrow \varphi_n = \phi_n$$

showing the uniqueness of the monic polynomial $\varphi_n \in \mathcal{P}_n$ satisfying (2).

Problem 4. (15 points) Consider the integral

$$I(f) := \int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} f(x)dx.$$

Derive a quadrature formula in the form

$$Q(f) := w_0 f(x_0) + w_1 f(x_1)$$

for this integral such that $Q(f) = I(f)$ for all $f \in \mathcal{P}_3$.

As shown in class, $Q(f) = I(f)\ \forall f \in \mathcal{P}_3$

if x_0, x_1 are the roots of $\varphi_2(x) \in \mathcal{P}_2$ s.t.

$$\int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} \varphi_2(x)\, p(x)\, dx = 0 \quad \forall p \in \mathcal{P}_1$$

Recall that $T_2(x) = 2x^2 - 1$ (Chebyshev polynomial of degree 2)

satisfies

$$\int_{-1} \frac{1}{\sqrt{1-\hat{x}^2}} T_2(\hat{x})\, p(\hat{x})\, d\hat{x} = 0 \quad \forall p \in \mathcal{P}_1$$

$\left(\begin{matrix}\text{letting}\\ x = 2\hat{x}\end{matrix}\right) \Longrightarrow$

$$\int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} T_2(x/2)\, p(x/2)\, \frac{1}{2}\, dx = 0 \quad \forall p \in \mathcal{P}_1$$

$$\Longrightarrow \int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} T_2(x/2)\, p(x)\, dx = 0 \quad \forall p \in \mathcal{P}_1.$$

Hence, $\varphi_2(x) = T_2(x/2) = \frac{x^2}{2} - 1$ with roots $x_0 = -\sqrt{2}$, $x_1 = \sqrt{2}$.

To find w_0, w_1, use Lagrange interpolation polynomial, i.e.,

$$w_0 = \int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} \frac{(x-\sqrt{2})}{(-2\sqrt{2})}\, dx \qquad w_1 = \int_{-2}^{2} \frac{1}{\sqrt{1-x^2/4}} \frac{(x+\sqrt{2})}{(2\sqrt{2})}\, dx$$

$\left(\begin{matrix}\text{letting}\\ x = 2\cos\theta\end{matrix}\right) =$

$$\int_{\pi}^{0} \frac{1}{\sqrt{1-\cos^2\theta}} \frac{2\cos\theta - \sqrt{2}}{(-2\sqrt{2})} (-2\sin\theta)\, d\theta \qquad = \int_{\pi}^{0} \frac{1}{\sqrt{1-\cos^2\theta}} \frac{2\cos\theta + \sqrt{2}}{(2\sqrt{2})} (-2\sin\theta)\, d\theta$$

$$= \pi \qquad\qquad = \pi$$

Hence,
$Q(f) =$
$\pi f(-\sqrt{2}) + \pi f(\sqrt{2})$

Problem 5. Let us consider the initial value problem

$$\begin{aligned} y' &= f(x, y), \\ y(x_0) &= y_0 \end{aligned} \tag{3}$$

where $y : [x_0, X_M] \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that this initial value problem has a unique solution y that is differentiable infinitely many times.

A one-step method can be defined for the solution of (3) and for a given positive integer N based on the update rule

$$y_{n+1} = y_n + hf\left(x_n + \frac{3h}{4},\, y_n + \frac{3h}{4} f(x_n, y_n)\right) \qquad n = 0, 1, 2, \ldots, N-1, \tag{4}$$

where $x_n := x_0 + nh$ and $h := (X_m - x_0)/N$.

(a) **(5 points)** Is the one-step method based on the update rule (4) consistent?

(b) **(15 points)** Determine the order of accuracy for this one-step method defined by (4).

(a) The update rule can be written as

$$y_{n+1} = y_n + h\,\phi(x_n, y_n; h)$$

where $\phi(x_n, y_n; h) = f(x_n + 3h/4,\, y_n + (3h/4) f(x_n, y_n))$.

As $\phi(x, y; 0) \equiv f(x, y)$, the method is consistent.

(b) Let us check the truncation error

$$(\ast)\quad T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f\left(x_n + \frac{3h}{4},\, y(x_n) + \frac{3h}{4} f(x_n, y(x_n))\right).$$

By Taylor's thm

$$y(x_{n+1}) = y(x_n) + y'(x_n)h + y''(x_n)\frac{h^2}{2} + O(h^3),$$

$$f\left(x_n + \frac{3h}{4},\, y(x_n) + \frac{3h}{4} f(x_n, y(x_n))\right) = f(x_n, y(x_n)) + \frac{3h}{4} f_x(x_n, y(x_n)) + \frac{3h}{4} f(x_n, y(x_n)) f_y(x_n, y(x_n)) + O(h^2)$$

Hence, order of accuracy for this method is 1

Plugging these in $(\ast)$

$$T_n = \left\{y'(x_n) + \frac{h}{2} y''(x_n) + O(h^2)\right\} - \left\{\underbrace{f(x_n, y(x_n))}_{y'(x_n)} + \frac{3h}{4}\overbrace{\left[f_x(x_n, y(x_n)) + f_y(x_n, y(x_n)) f(x_n, y(x_n))\right]}^{y''(x_n)} + O(h^2)\right\}$$

$$= -\frac{h}{4} y''(x_n) + O(h^2).$$

Problem 6. (15 points) Consider the ∞-norm

$$\|f\|_\infty := \max_{x\in[-1,1]} |f(x)| \tag{5}$$

defined over $C[-1,1]$, the space of continuous functions on $[-1,1]$.

Find the polynomial $p_* \in \mathcal{P}_2$ such that

$$\|(2x^3+x^2) - p_*\|_\infty = \min_{p\in\mathcal{P}_2} \|(2x^3+x^2) - p\|_\infty$$

where $\|\cdot\|_\infty$ is defined as in (5).

Recall that $\tilde{p}_*(x) = x^{n+1} - 2^{-n} T_{n+1}(x) \in \mathcal{P}_n$ is such that (Here T_{n+1} is the Chebyshev polynomial of degree $n+1$)

$$\|x^{n+1} - \tilde{p}_*\|_\infty \leq \|x^{n+1} - p_n\|_\infty \quad \forall p_n \in \mathcal{P}_n.$$

in particular (for $n=2$) $\hat{p}_*(x) = x^3 - (1/4)T_3(x)$

$$= x^3 - (1/4)(4x^3 - 3x) = 3x/4$$

satisfies

$$\|x^3 - \hat{p}_*\|_\infty \leq \|x^3 - p_2\|_\infty \quad \forall p_2 \in \mathcal{P}_2$$

$$\Longrightarrow \|2x^3 - 2\hat{p}_*\|_\infty \leq \|2x^3 - 2p_2\|_\infty \quad \forall p_2 \in \mathcal{P}_2$$

$$\Longrightarrow \|(2x^3+x^2) - (2\hat{p}_* + x^2)\|_\infty \leq \|(2x^3+x^2) - (2p_2+x^2)\|_\infty \quad \forall p_2 \in \mathcal{P}_2$$

Letting $\boxed{p_*(x) := 2\hat{p}_*(x) + x^2 = x^2 + 3x/2}$, as $2p_2 + x^2$ on the right in the last inequality can be any polynomial in $\mathcal{P}_2$, we have

$$\|(2x^3+x^2) - p_*\|_\infty \leq \|(2x^3+x^2) - p\|_\infty \quad \forall p \in \mathcal{P}_2.$$