MATH 305: Numerical Anaysis

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Fall Semester 2018
1st Midterm Examination

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NAME ________________

STUDENT ID ________________

SIGNATURE ________________

- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 100 minutes.

Problem 1 Given $a \in \mathbb{R}$, $a \neq 0$, one way of computing $1/a$ without performing any division is the sequence $\{x_k\}$ in $\mathbb{R}$ defined by

$$x_{k+1} = (a+1)x_k - 1$$

for a given $x_0 \in \mathbb{R}$.

(a) **(13 points)** Determine the values of a such that $\{x_k\}$ converges to $1/a$ for all $x_0 \in \left(\frac{1}{a} - \delta, \frac{1}{a} + \delta\right)$ for some $\delta > 0$. Explain your reasoning.

(b) **(12 points)** Determine the values of a such that $\{x_k\}$ cannot converge to $1/a$ unless $x_0 = 1/a$. Explain why.

Problem 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(4)}(x)$ is continuous on $\mathbb{R}$. Furthermore, let $x_* \in \mathbb{R}$ satisfy $f(x_*) = f'(x_*) = 0$, $f''(x_*) \neq 0$, $f'''(x_*) \neq f''(x_*)$.

Suppose that the sequence $\{x_k\}$ in $\mathbb{R}$ defined by

$$x_{k+1} \quad = \quad x_k - \frac{f'(x_k) + f(x_k)}{f''(x_k)}$$

for a given $x_0 \in \mathbb{R}$ converges to x_*. Show that the order of this convergence to x_* is two.

Problem 3 For a given invertible matrix $A \in \mathbb{R}^{n\times n}$, let us consider the function

$$\mathbf{A.} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{A.}(b) := Ab.$$

(a) **(15 points)** Suppose that $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$. Denote with $\|A\|, \|A^{-1}\|$ the matrix norms of A, A^{-1}, respectively, induced by $\|\cdot\|$ on $\mathbb{R}^n$. Show that, at every $\widehat{b} \in \mathbb{R}^n$, the relative local condition number of **A.** at $\widehat{b}$ (defined over $D = \mathbb{R}^n$) satisfies

$$\operatorname{cond}_{\widehat{b}} \mathbf{A.} \quad \leq \quad \|A\|\|A^{-1}\|.$$

(b) **(10 points)** Now consider part (a) with the particular choice of $\|\cdot\| = \|\cdot\|_1$ on $\mathbb{R}^n$ (that is, $\|\cdot\|$ is the 1-norm on $\mathbb{R}^n$), and the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$, that is the matrix 1-norm on $\mathbb{R}^{n\times n}$.

Find a particular $\widehat{b}$ so that

$$\operatorname{cond}_{\widehat{b}} \mathbf{A.} \quad = \quad \|A\|_1\|A^{-1}\|_1.$$

Problem 4 Show that every matrix $A \in \mathbb{R}^{n\times n}$ has a factorization of the form

$$AP = LU$$

where $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix, and $U \in \mathbb{R}^{n\times n}$ is a unit upper triangular matrix (that is U is an upper triangular matrix whose every diagonal entry is 1).