Solutions

MATH 305: Numerical Anaysis

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Fall Semester 2018
1st Midterm Examination

NAME ______________

STUDENT ID ______________

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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 100 minutes.

Problem 1 Given $a \in \mathbb{R}$, $a \neq 0$, one way of computing $1/a$ without performing any division is the sequence $\{x_k\}$ in $\mathbb{R}$ defined by

$$x_{k+1} = (a+1)x_k - 1$$

for a given $x_0 \in \mathbb{R}$.

(a) **(13 points)** Determine the values of a such that $\{x_k\}$ converges to $1/a$ for all $x_0 \in \left(\frac{1}{a} - \delta, \frac{1}{a} + \delta\right)$ for some $\delta > 0$. Explain your reasoning.

(b) **(12 points)** Determine the values of a such that $\{x_k\}$ cannot converge to $1/a$ unless $x_0 = 1/a$. Explain why.

The sequence $\{x_k\}$ if it converges must converge to the unique fixed point $1/a$ of $g(x) = (a+1)x - 1$.

(a) As $g'(x)$ is continuous, $\{x_k\}$ ~~with~~ converges to the unique fixed point for all x_0 close to this fixed point $1/a$ if

$$|g'(1/a)| = |a+1| < 1$$

$$\iff -2 < a < 0.$$

(b) Again as $g'(x)$ is continuous, $\{x_k\}$ cannot converge to the fixed point $1/a$ unless $x_0 = 1/a$ if

$$|g'(1/a)| = |a+1| > 1$$

$$\iff a > 0 \quad \text{or} \quad a < -2.$$

Problem 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(4)}(x)$ is continuous on $\mathbb{R}$. Furthermore, let $x_* \in \mathbb{R}$ satisfy $f(x_*) = f'(x_*) = 0$, $f''(x_*) \neq 0$, $f'''(x_*) \neq f''(x_*)$.

Suppose that the sequence $\{x_k\}$ in $\mathbb{R}$ defined by

$$x_{k+1} = x_k - \frac{f'(x_k) + f(x_k)}{f''(x_k)}$$

for a given $x_0 \in \mathbb{R}$ converges to x_*. Show that the order of this convergence to x_* is two.

Letting $g(x) = x - \dfrac{f'(x) + f(x)}{f''(x)}$ it suffices to show that

$$g'(x_*) = 0 \quad \text{and} \quad g''(x_*) \neq 0.$$

in order to deduce an order of convergence of two for $\{x_k\}$.

$$g'(x) = 1 - \frac{f''(x) + f'(x)}{f''(x)} + \frac{f'(x) + f(x)}{f''(x)^2} f'''(x)$$

$$\Rightarrow g'(x_*) = 1 - \frac{f''(x_*)}{f''(x_*)} = 0.$$

$$g''(x) = -\frac{f'''(x) + f''(x)}{f''(x)} + \frac{f''(x) + f'(x)}{f''(x)^2} f'''(x) + \frac{f''(x) + f'(x)}{f''(x)^2} f'''(x)$$
$$+ \frac{f'(x) + f(x)}{f''(x)^2} f^{(4)}(x) - 2\frac{f'(x) + f(x)}{f''(x)^3} f'''(x)^2$$

$$\Rightarrow g''(x_*) = -1 + \frac{f'''(x_*)}{f''(x_*)} \neq 0 \quad \text{(since } f'''(x_*) \neq f''(x_*)\text{)}$$

By Taylor's thm with 2nd order remainder

$$|x_{k+1} - x_*| = |g(x_k) - g(x_*)| = \left| g'(x_*)(x_k - x_*) + \frac{g''(\xi_k)}{2}(x_k - x_*)^2 \right|$$
$$= \left| \frac{g''(\xi_k)}{2} \right| |x_k - x_*|^2 \quad \exists\, \xi_k \text{ in the open interval with end-points } x_k, x_*$$

Taking the limit as $k \to \infty$, as $g''(x)$ is continuous, $\displaystyle\lim_{k \to \infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} = \frac{|g''(x_*)|}{2} \neq 0.$

Question 3 For a given invertible matrix $A \in \mathbb{R}^{n\times n}$, let us consider the function

$$\mathbf{A.} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{A.}(b) := Ab.$$

(a) **(15 points)** Suppose that $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$. Denote with $\|A\|, \|A^{-1}\|$ the matrix norms of A, A^{-1}, respectively, induced by $\|\cdot\|$ on $\mathbb{R}^n$. Show that, at every $\widehat{b} \in \mathbb{R}^n$, the relative local condition number of $\mathbf{A.}$ at $\widehat{b}$ (defined over $D = \mathbb{R}^n$) satisfies

$$\operatorname{cond}_{\widehat{b}} \mathbf{A.} \le \|A\|\|A^{-1}\|.$$

(b) **(10 points)** Now consider part (a) with the particular choice of $\|\cdot\| = \|\cdot\|_1$ on $\mathbb{R}^n$ (that is, $\|\cdot\|$ is the 1-norm on $\mathbb{R}^n$), and the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$, that is the matrix 1-norm on $\mathbb{R}^{n\times n}$.

Find a particular $\widehat{b}$ so that

$$\operatorname{cond}_{\widehat{b}} \mathbf{A.} = \|A\|_1\|A^{-1}\|_1.$$

(a) Absolute local condition number

$$\operatorname{Cond}_{\hat{b}} A. = \sup_{\delta b,\, \delta b \neq 0} \frac{\|A.(\hat{b}+\delta b) - A.(\hat{b})\|}{\|\delta b\|}$$

$$= \sup_{\delta b,\, \delta b \neq 0} \frac{\|A\,\delta b\|}{\|\delta b\|} = \|A\|$$

Relative local condition number

$$\operatorname{cond}_{\hat{b}} A. = \frac{\|\hat{b}\|}{\|A.(\hat{b})\|} \operatorname{Cond}_{\hat{b}} A.$$

$$= \frac{\|\hat{b}\|\,\|A\|}{\|A\hat{b}\|}$$

By the submultiplicative property
$\|\hat{b}\| = \|A^{-1}A\hat{b}\| \leqslant \|A^{-1}\|\,\|A\hat{b}\|$.

Hence,

$$\operatorname{cond}_{\hat{b}} A. \leqslant \frac{\|A^{-1}\|\,\|A\hat{b}\|\,\|A\|}{\|A\hat{b}\|} = \|A^{-1}\|\,\|A\|$$

as desired.

(b) Now suppose $\|\cdot\| = \|\cdot\|_1$.

In the derivation in part (a) if

$$\|\hat{b}\|_1 = \|A^{-1}A\hat{b}\|_1$$
$$= \|A^{-1}\|_1 \|A\hat{b}\|_1$$

then we have

$$\text{cond}_{\hat{b}} A. = \|A^{-1}\|_1 \|A\|_1 .$$

Hence, we need to find a $\hat{b}$ such that

$$\|A^{-1}A\hat{b}\|_1 = \|A^{-1}\|_1 \|A\hat{b}\|_1$$

Letting $\hat{v} := A\hat{b}$, we have

$$\|A^{-1}\hat{v}\|_1 = \|A^{-1}\|_1 \|\hat{v}\|_1 \quad \left(\text{equivalently } \frac{\|A^{-1}\hat{v}\|_1}{\|\hat{v}\|_1} = \|A^{-1}\|_1 \right)$$

if $\hat{v} = e_k$ (or any multiple of e_k)

where $\underset{\in\{1,\dots,n\}}{k}$ is such that $\|A^{-1}(:,k)\|_1 = \max\limits_{j=1,\dots,n} \|A^{-1}(:,j)\|_1$.

This yields $\hat{b} = A^{-1}v = A^{-1}e_k = A^{-1}(:,k)$.

Question 4 Show that every matrix $A \in \mathbb{R}^{n\times n}$ has a factorization of the form

$$(+) \qquad AP = LU$$

where $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix, and $U \in \mathbb{R}^{n\times n}$ is a unit upper triangular matrix (that is U is an upper triangular matrix whose every diagonal entry is 1).

The proof is by induction on n.

For $n=1$, (+) holds trivially as

$$\underbrace{A}\cdot \underset{P}{1} = A \cdot \underset{L}{1} \cdot \underset{U}{1}.$$

Now for an arbitrary $n \geq 2$, suppose all $(n-1)\times(n-1)$ matrices have factorizations of the form (+). We claim

$$(++) \qquad \left(\begin{bmatrix} a_{1r} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} =\right) A P_{1r} = \begin{bmatrix} b & 0 \\ \alpha & C \end{bmatrix}\begin{bmatrix} 1 & \beta^T \\ 0 & I_{n-1}\end{bmatrix}$$

for some $b \in \mathbb{R}$, $\beta, \alpha \in \mathbb{R}^{n-1}$ and $C \in \mathbb{R}^{(n-1)\times(n-1)}$.

Here, P_{1r} is the permutation matrix obtained from I_n by interchanging its 1st and rth columns, where $\underset{\in\{1,\dots,n\}}{r}$ is s.t. $|a_{1r}| = \max_{j=1,\dots,n} |a_{1j}|$.

If $a_{1r}=0$, then $A_{12}=0$ and $AP_{1r} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & I_{n-1}\end{bmatrix}$

So a factorization of the form (++) exists.

Otherwise, (++) holds if

$$a_{1r} = b, \quad A_{21} = \alpha$$
$$A_{12} = b\beta^T, \quad A_{22} = \alpha\beta^T + C.$$

Hence, we set $b = a_{1r}$, $\alpha = A_{21}$, $\beta = A_{12}^T/b$ and $C = A_{22} - A_{21}A_{12}/b$ so that (++) holds.

By the inductive hypothesis C has a factorization

$$(+++) \quad C\hat{P} = \hat{L}\hat{U}$$

for some permutation matrix $\hat{P} \in \mathbb{R}^{(n-1)\times(n-1)}$, lower triangular matrix $\hat{L} \in \mathbb{R}^{(n-1)\times(n-1)}$ and unit upper triangular matrix $\hat{U} \in \mathbb{R}^{(n-1)\times(n-1)}$.

Now exploiting (++) and (+++)

$$AP_{1r} = \begin{bmatrix} b & 0 \\ \alpha & \underbrace{C\hat{P}}_{\hat{L}\hat{U}}\hat{P}^T \end{bmatrix} \begin{bmatrix} 1 & \beta^T \\ 0 & I_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} b & 0 \\ \alpha & \hat{L} \end{bmatrix} \begin{bmatrix} 1 & \beta^T \\ 0 & \hat{U}\hat{P}^T \end{bmatrix}$$

$$= \begin{bmatrix} b & 0 \\ \alpha & \hat{L} \end{bmatrix} \begin{bmatrix} 1 & \beta^T\hat{P} \\ 0 & \hat{U} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}^T \end{bmatrix}$$

$$\Rightarrow \underbrace{AP_{1r}\begin{bmatrix} 1 & 0 \\ 0 & \hat{P} \end{bmatrix}}_{\substack{P \in \mathbb{R}^{n\times n} \\ \text{permutation} \\ \text{matrix}}} = \underbrace{\begin{bmatrix} b & 0 \\ \alpha & \hat{L} \end{bmatrix}}_{\substack{L \in \mathbb{R}^{n\times n} \\ \text{lower} \\ \text{triangular}}} \underbrace{\begin{bmatrix} 1 & \beta^T\hat{P} \\ 0 & \hat{U} \end{bmatrix}}_{\substack{U \in \mathbb{R}^{n\times n} \\ \text{unit upper} \\ \text{triangular}}},$$

and the existence of the factorization (+) follows from induction.