MATH 305: Numerical Anaysis

Instructor: Emre Mengi

Fall Semester 2018
2nd Midterm Examination

NAME ____________________

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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 110 minutes.

Problem 1

(a) **(15 points)** Derive a quadrature formula in the form

$$Q(f) \ := \ w_0 f(1) + w_1 f(2)$$

for the integral

$$I(f) \ = \ \int_0^3 f(x)dx \tag{1}$$

such that $Q(f) = I(f)$ whenever $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree at most one.

(b) **(10 points)** Now let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f''(x)$ exists and continuous on $\mathbb{R}$, and let $M_2 := \max_{x \in [0,3]} |f''(x)|$.

Show that the quadrature formula that you derived in part (a) satisfies

$$|I(f) - Q(f)| \leq \frac{11 M_2}{12},$$

or otherwise a similar upper bound, where $I(f)$ is the integral in (1).

Problem 2 Consider the sequence $\{q^{(k)}\}$ defined by $q^{(k+1)} := Aq^{(k)}/\|Aq^{(k)}\|_2$ and $q^{(0)} := (1,0)^T$ for

$$A = \begin{bmatrix} 1 & -8 \\ -8 & -11 \end{bmatrix}.$$

(a) **(13 points)** Determine a unit vector v_* such that $\|q^{(k)} - c_k v_*\|_2 \to 0$ as $k \to \infty$ for some sequence $\{c_k\}$ in $\mathbb{R}$ satisfying $\lim_{k\to\infty} |c_k| = 1$.

(b) **(12 points)** Find the limit

$$\lim_{k\to\infty} \frac{\|q^{(k+1)} - c_{k+1}v_*\|_2}{\|q^{(k)} - c_k v_*\|_2},$$

where $\{c_k\}$, $\{q^{(k)}\}$, v_* are as in part (a).

Problem 3 Suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ has continuous first derivatives, and x_* is a fixed point of g such that

$$\left| \frac{\partial g_j(x_*)}{\partial x_\ell} \right| < \frac{1}{5}$$

for $j = 1, 2$ and $\ell = 1, 2$.

Show that there exists an $\varepsilon > 0$ such that for all $x^{(0)} \in \overline{B}_\varepsilon(x_*)$ the sequence $\{x^{(k)}\},\ x^{(k+1)} = g(x^{(k)})$ converges to x_* and satisfies

$$\frac{\|x^{(k+1)} - x_*\|_\infty}{\|x^{(k)} - x_*\|_\infty} < \frac{1}{2}$$

for all k.

(*Recall that* $\overline{B}_\varepsilon(x_*) := \{x \in \mathbb{R}^2 \mid \|x - x_*\|_\infty \leq \varepsilon\}$.)

Problem 4 A matrix $H \in \mathbb{R}^{n\times n}$ is called Hessenberg if $h_{jk} = 0$ whenever $j - k > 1$. For instance the following 4×4 matrix is Hessenberg.

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 4 & 5 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

For every Hessenberg matrix $H \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QH = R \tag{2}$$

for some upper triangular matrix $R \in \mathbb{R}^{n\times n}$.

Given a Hessenberg matrix $H \in \mathbb{R}^{n\times n}$, design an algorithm that computes an upper triangular matrix $R \in \mathbb{R}^{n\times n}$ satisfying (2) for some orthogonal matrix $Q \in \mathbb{R}^{n\times n}$. Your algorithm must perform at most $\sim cn^2$ arithmetic operations for some constant c (that is independent of n).