

Solutions

MATH 305: Numerical Analysis

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Fall Semester 2018
2nd Midterm Examination

NAME _____

STUDENT ID _____

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- Put your name, student ID and signature in the spaces provided above.
- Duration for this exam is 110 minutes.

Problem 1

- (a) (15 points) Derive a quadrature formula in the form

$$Q(f) := w_0 f(1) + w_1 f(2)$$

for the integral

$$I(f) = \int_0^3 f(x) dx \quad (1)$$

such that $Q(f) = I(f)$ whenever $f : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial of degree at most one.

- (b) (10 points) Now let
- $f : \mathbb{R} \rightarrow \mathbb{R}$
- be a function such that
- $f''(x)$
- exists and continuous on
- \mathbb{R}
- , and let
- $M_2 := \max_{x \in [0,3]} |f''(x)|$
- .

Show that the quadrature formula that you derived in part (a) satisfies

$$|I(f) - Q(f)| \leq \frac{11M_2}{12},$$

or otherwise a similar upper bound, where $I(f)$ is the integral in (1).

$$(a) \quad Q(f) = \int_0^3 p_1(x) dx$$

where p_1 is the Lagrange interpolation polynomial through $(1, f(1))$ and $(2, f(2))$.

$$\begin{aligned} Q(f) &= \int_0^3 \left(\frac{x-2}{1-2} \right) f(1) + \left(\frac{x-1}{2-1} \right) f(2) dx \\ &= \left[\int_0^3 (2-x) dx \right] f(1) + \left[\int_0^3 (x-1) dx \right] f(2) \\ &= \frac{3}{2} f(1) + \frac{3}{2} f(2) \end{aligned}$$

(b) By the standard result regarding the interpolation error

$$|I(f) - Q(f)| = \left| \int_0^3 f(x) - p_1(x) dx \right|$$

$$(+) \leq \int_0^3 |f(x) - p_1(x)| dx \leq \int_0^3 \frac{M_2}{2} |(x-1)(x-2)| dx$$

where

$$\int_0^3 |(x-1)(x-2)| dx = \int_0^1 (x-1)(x-2) dx - \int_1^2 (x-1)(x-2) dx + \int_2^3 (x-1)(x-2) dx$$

$$= \{F(1) - F(0)\} - \{F(2) - F(1)\} + \{F(3) - F(2)\}$$

$$= F(3) - 2F(2) + 2F(1) + F(0) = \frac{3}{2} - \frac{4}{3} + \frac{10}{6} = 1\frac{1}{6}$$

for $F(x) = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x$

Hence, from (+), we have

$$|I(f) - Q(f)| \leq \frac{M_2}{12}$$

Problem 2 Consider the sequence $\{q^{(k)}\}$ defined by $q^{(k+1)} := Aq^{(k)} / \|Aq^{(k)}\|_2$ and $q^{(0)} := (1, 0)^T$ for

$$A = \begin{bmatrix} 1 & -8 \\ -8 & -11 \end{bmatrix}.$$

(a) (13 points) Determine a unit vector v_* such that $\|q^{(k)} - c_k v_*\|_2 \rightarrow 0$ as $k \rightarrow \infty$ for some sequence $\{c_k\}$ in \mathbb{R} satisfying $\lim_{k \rightarrow \infty} |c_k| = 1$.

(b) (12 points) Find the limit

$$\lim_{k \rightarrow \infty} \frac{\|q^{(k+1)} - c_{k+1} v_*\|_2}{\|q^{(k)} - c_k v_*\|_2},$$

where $\{c_k\}$, $\{q^{(k)}\}$, v_* are as in part (a).

(a) The sequence is the one generated by the power iteration, so v_* is any unit eigenvector corresponding to the largest eigenvalue in $|\cdot|$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & -8 \\ -8 & -11-\lambda \end{pmatrix} = (1-\lambda)(-11-\lambda) - 64 \\ &= \lambda^2 + 10\lambda - 75 = (\lambda + 15)(\lambda - 5) \end{aligned}$$

so the eigenvalues are $\lambda_1 = -15$ and $\lambda_2 = 5$.

corresponding eigenvectors

$$\lambda_1 = -15$$

$$(A + 15I)v_1 = \begin{bmatrix} 16 & -8 \\ -8 & 4 \end{bmatrix} v_1 = 0$$

$$\Rightarrow v_1 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for any scalar } c_1 \neq 0$$

$$\lambda_2 = 5$$

$$(A - 5I)v_2 = \begin{bmatrix} -4 & -8 \\ -8 & -16 \end{bmatrix} v_2 = 0$$

$$\Rightarrow v_2 = c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for any scalar } c_2 \neq 0$$

$$\text{Hence, } v_* = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) As $\{q^{(k)}\}$ is the sequence by the power iteration

$$\lim_{k \rightarrow \infty} \frac{\|q^{(k+1)} - c_{k+1} v_*\|_2}{\|q^{(k)} - c_k v_*\|_2} = \left| \frac{\lambda_2}{\lambda_1} \right| = \frac{1}{3}.$$

Problem 3 Suppose $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has continuous first derivatives, and x_* is a fixed point of g such that

$$\left| \frac{\partial g_j(x_*)}{\partial x_\ell} \right| < \frac{1}{5}$$

for $j = 1, 2$ and $\ell = 1, 2$.

Show that there exists an $\varepsilon > 0$ such that for all $x^{(0)} \in \overline{B}_\varepsilon(x_*)$ the sequence $\{x^{(k)}\}$, $x^{(k+1)} = g(x^{(k)})$ converges to x_* , and satisfies

$$(+)\quad \frac{\|x^{(k+1)} - x_*\|_\infty}{\|x^{(k)} - x_*\|_\infty} < \frac{1}{2}$$

for all k .

(Recall that $\overline{B}_\varepsilon(x_*) := \{x \in \mathbb{R}^2 \mid \|x - x_*\|_\infty \leq \varepsilon\}$.)

By continuity of first derivatives of g , $\exists \varepsilon > 0$ such that

$$\left| \frac{\partial g_j(x)}{\partial x_\ell} \right| < \frac{1}{4} \quad \forall x \in \overline{B}_\varepsilon(x_*)$$

for $j = 1, 2$ and $\ell = 1, 2$.

Suppose $x^{(k)} \in \overline{B}_\varepsilon(x_*)$, we show the satisfaction of (+). To this end, first observe $x^{(k+1)} - x_* = g(x^{(k)}) - g(x_*)$, hence define $\phi_j: \mathbb{R} \rightarrow \mathbb{R}$, $\phi_j(\alpha) := g_j(x_* + \alpha(x^{(k)} - x_*))$ for $j = 1, 2$. Then, we have

$$\begin{aligned} g_j(x^{(k)}) - g_j(x_*) &= \phi_j(1) - \phi_j(0) \stackrel{\text{MVT}}{=} \phi_j'(\eta) \quad \exists \eta \in (0, 1) \\ &= \sum_{\ell=1}^2 \frac{\partial g_j(x_* + \eta(x^{(k)} - x_*))}{\partial x_\ell} \{x^{(k)} - x_*\}_\ell \end{aligned}$$

$$\begin{aligned} \text{implying} \quad |g_j(x^{(k)}) - g_j(x_*)| &\leq \sum_{\ell=1}^2 \left| \frac{\partial g_j(x_* + \eta(x^{(k)} - x_*))}{\partial x_\ell} \right| \|x^{(k)} - x_*\|_\infty \\ &< \frac{1}{2} \|x^{(k)} - x_*\|_\infty, \quad j = 1, 2. \end{aligned}$$

Hence, we deduce

$$\begin{aligned} \|x^{(k+1)} - x_*\|_\infty &= \|g(x^{(k)}) - g(x_*)\|_\infty \leq |g_j(x^{(k)}) - g_j(x_*)| \\ &< \frac{1}{2} \|x^{(k)} - x_*\|_\infty \end{aligned}$$

as desired.

Now it follows from (+) and by induction that if $x^{(0)} \in \overline{B}_\varepsilon(x_*)$, then $x^{(k)} \in \overline{B}_\varepsilon(x_*) \quad \forall k$. Furthermore,

$$\lim_{k \rightarrow \infty} \|x^{(k)} - x_*\|_\infty = \lim_{k \rightarrow \infty} \left(\frac{1}{2}\right)^k \|x^{(0)} - x_*\|_\infty = 0 \Rightarrow \lim_{k \rightarrow \infty} x^{(k)} = x_*$$

as desired.

Now it follows by induction that if $x^{(0)} \in \bar{B}_\varepsilon(x_*)$, then $x^{(k)} \in \bar{B}_\varepsilon(x_*)$ and (+) holds for all k .

Finally,

$$\lim_{k \rightarrow \infty} \|x^{(k)} - x_*\|_\infty = \lim_{k \rightarrow \infty} \left(\frac{1}{2}\right)^k \|x^{(0)} - x_*\|_\infty = 0$$

$$\implies \lim_{k \rightarrow \infty} x^{(k)} = x_*$$

as desired. □

Problem 4 A matrix $H \in \mathbb{R}^{n \times n}$ is called Hessenberg if $h_{jk} = 0$ whenever $j - k > 1$. For instance the following 4×4 matrix is Hessenberg.

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 4 & 5 & 2 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

For every Hessenberg matrix $H \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$QH = R \quad (2)$$

for some upper triangular matrix $R \in \mathbb{R}^{n \times n}$.

Given a Hessenberg matrix $H \in \mathbb{R}^{n \times n}$, design an algorithm that computes an upper triangular matrix $R \in \mathbb{R}^{n \times n}$ satisfying (2) for some orthogonal matrix $Q \in \mathbb{R}^{n \times n}$. Your algorithm must perform at most $\sim cn^2$ arithmetic operations for some constant c (that is independent of n).

Reflectors or rotators can be used.
Let us use the reflectors, and proceed column by column from left to right.

kth step

(k, k) entry

$(k-1) \times (k-1)$ upper triangular

$A^{(k)}$ $B^{(k)}$

To be modified

reflector H_k based on $v^{(k)}$ - left multiply

$H_k = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & I_2 - q^{(k)} q^{(k)T} & 0 \\ 0 & 0 & I_{n-k-2} \end{bmatrix}$ where $q^{(k)} = \frac{v^{(k)} + \text{sign}(v_1^{(k)}) \|v^{(k)}\|_2 e_1}{\|v^{(k)} + \text{sign}(v_1^{(k)}) \|v^{(k)}\|_2 e_1\|_2}$

Pseudocode

for $k = 1, \dots, n-2$ (nothing to do on the $(n-1)$ th and n th columns)

$v \leftarrow A(k:k+1, k)$

$q \leftarrow v + \text{sign}(v_1) \|v\|_2 e_1$

$A(k:k+1, k:n) \leftarrow A(k:k+1, k:n) \textcircled{3} 2q \textcircled{2} q^T \textcircled{1} A(k:k+1, k:n)$

end

arithmetic operations

$$\textcircled{1} \quad \sim 3(n-k)$$

$$\textcircled{2} \quad \sim 2(n-k)$$

$$\textcircled{3} \quad \sim 2(n-k)$$

Total

$$\sim \sum_{k=1}^{n-2} 7(n-k) \sim 7 \sum_{k=1}^n k$$

$$\sim 7n^2/2$$